Series on Mathematics Education Vol. 5

Edited by
Alexander Karp • Bruce R. Vogeli

RUSSIAN MATHEMATICS EDUCATION

Programs and Practices

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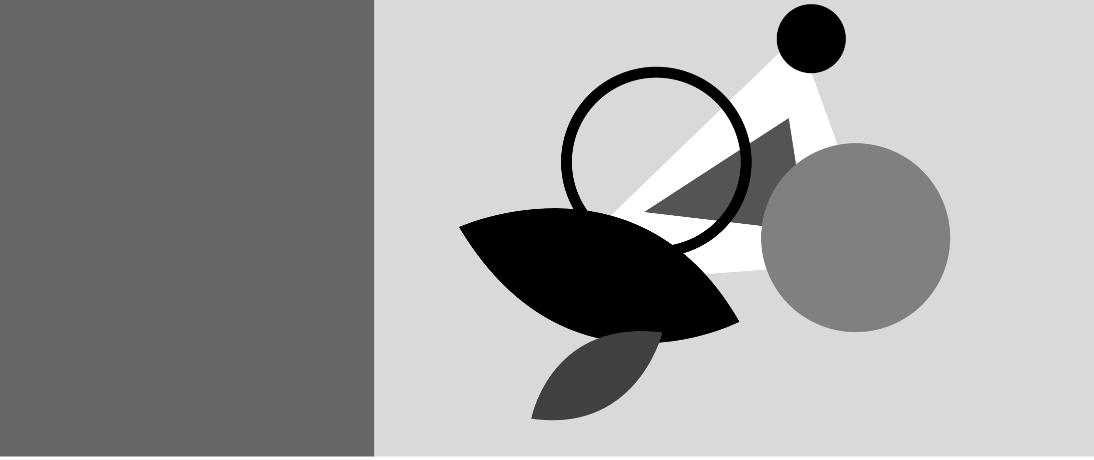

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Alexander Karp

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Columbia University, USA

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Contents

Introduction

This volume is a continuation of the previously published work *Russian Mathematics Education: History and World Significance*. As its title indicates, its primary focus is on Russian programs and practices in school mathematics education. Thus, it deals mainly with the contemporary situation, although this does not rule out a historical perspective, without which it is often impossible to understand what is happening today. Practices that are widespread and established at the time of the book's publication may change in the near future. More profound characteristics, positions, and traditions, however, do not change quickly: the aim of this volume is to help readers to become acquainted with them and to understand them.

These traditions, however, may be understood in different ways. More precisely, it may be said that a genuine understanding of what has happened and what is happening in Russian mathematics education requires a recognition of the fact that Russian mathematics education includes different traditions and different perspectives on these traditions. The editors of these two volumes have strived to represent this variety of perspectives. Thus, invited contributors include well-known figures in Russian education, the authors of widely used and sometimes competing textbooks, as well as mathematics educators who are currently working outside of Russia.

The chapters in this volume are devoted to different aspects of mathematics education in Russia and to different processes taking place in it. First chapter by Alexander Karp and Leonid Zvavich discusses mathematics lessons and the traditional approaches to structuring mathematics lessons in Russia. This chapter also contains basic information about the Russian system of mathematics education that may be useful to the readers.

Mathematical subjects and courses taught in Russian schools are addressed in special chapters in this volume. Olga Ivashova analyzes the elementary school mathematics program in chapter two. The next chapter, by Alexander Karp and Alexey Werner, is devoted to the course in geometry — that is distributed over five years in Russia (USSR), in contrast to many other countries. Liudmila Kuznetsova, Elena Sedova, Svetlana Suvorova, Saule Troitskaya discuss the teaching of algebra; Mikhael Jackubson describes instruction in elementary calculus (which is a required course for all students in the higher grades); and Evgeny Bunimovich addresses the teaching of topics that are new to Russian schools — combinatorics, probability, and statistics.

Subsequent chapters are devoted to the structures and systems in Russian mathematics education, important for students of different ages and for the teaching of different mathematical subjects. Alexander Karp traces the history, practices, and distinctive features of so-called *schools with an advanced course of study in mathematics* and *schools specializing in the humanities*, which are of relatively recent provenance. The next chapter, written by Alexander Karp and Leonid Zvavich, is devoted to mathematics assessment in Russian schools and the chapter that follows it, written by Albina Marushina and Maksim Pratusevich, addresses extracurricular work in mathematics.

Finally, the last chapter, written by Alexander Karp and Roza Leikin, differs somewhat from the preceding ones. Its subject is not the school itself, but academic studies devoted to mathematics education. The authors characterize the directions, goals, and styles of academic studies in mathematics education in Russia over the last twenty years (mainly by analyzing dissertational studies).

As in the first volume, several chapters were originally written in Russian and subsequently translated into English. The editors wish to thank Ilya Bernstein and Sergey Levchin for help in preparing the manuscript for publication. The editors also express their gratitude to Heather Gould and Gabriella Oldham for their assistance in proofreading manuscripts.

1
On the Mathematics Lesson

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1 Introduction

The basic form of mathematics instruction in Russia is the classroom lesson. Of course, other forms exist, such as individual instruction, which is used with poorly performing students after classes or with seriously ill children at home. Naturally, teaching also takes place outside of class — through extracurricular activities, homework, and so on — so it cannot be equated with that which goes on in class. Nonetheless, it would be no mistake to repeat that the basic form of mathematics instruction is the classroom lesson. Not for nothing were those who in Soviet times were most concerned with teaching students, and not with what was conceived of as communist character-building work, contemptuously labeled "lesson providers."

Every day in the upper grades, six or seven required classes are followed by optional activities outside of the standard schedule: elective classes, and special courses in various subjects, including those not part of the standard school program — in other words, effectively more classes (but, in contrast to those that are part of the standard schedule, these are not mandatory for everyone). Over the course of his or her schooling, a student attends about 2000 mathematics classes, while a mathematics teacher teaches several tens of thousands of classes throughout his or her career (Ryzhik, 2003). Consequently, much has been written and discussed about planning and conducting classes, in all subjects in general and in mathematics in particular. Dozens of manuals on conducting classes have been developed and published, presenting problems for solving in class, quizzes for testing students in class, and simply lesson plans. Even today, despite the availability of numerous publications and possibilities for copying necessary materials, lectures in which an experienced teacher presents and discusses various approaches to conducting lessons remain popular.

This chapter is devoted to the lesson and how it is constructed and conducted in Russian mathematics classrooms. Of course, it is impossible to talk about any system for conducting classes in mathematics that is common to all Russian (Soviet) teachers: the country is large, and although the same requirements apply everywhere and control has sometimes been very rigid, the diversity of the lessons has been and remains great. Sometimes, lessons conducted in accordance with official requirements have been very successful; on other occasions, although they apparently followed the rules, some lessons have clearly turned out badly. Additionally, analyses of lessons conducted by mathematics supervisors even in Stalin's time include numerous remarks suggesting that classes were not conducted according to the requirements. Nonetheless, the very existence of common requirements leads us to reflect on some common characteristics of Russian mathematics lessons. Many of these characteristics emerged during the 1930s–1950s — the formative years of Soviet schools — after almost all post-Revolution explorations were rejected. We will therefore discuss the methodological works of this period, gradually progressing into modern times. But first, to provide some background, we will say a few words about the conditions under which classes are conducted today.

2 Who Participates in the Class and Where Classes Are Conducted: Background

To provide a better understanding of the specific character of Russian classes, we must describe certain important features of the way in which the teaching process has been organized in Russia, both traditionally and at the present time.

2.1 *Teachers and Students*

Perhaps the most important difference between the teaching of mathematics in Russia and, say, in the United States is the fact that usually a teacher works with the same class for a considerable length of time — the composition of the class virtually does not change, and the class continues to have the same teacher. Instruction is broken down not into different courses that the students can take, but simply into different years of schooling — in fifth grade everyone studies specific topics, and in sixth grade everyone moves on to other topics. A teacher can be assigned to a fifth-grade classroom and, in principle, remain with the students until their graduation (note that in Russia there is no distinction between middle and high school in the sense that students of all ages study in the same building, have the same principal, and so on).

One of the authors of this chapter, for example, had the same mathematics teacher during all of his years in school, from fifth grade until his final year (which, at the time, was tenth grade). The composition of the class did not change much either. Of course, there were "new kids" who would come from other schools, usually because their families had moved. And, of course, some students left, usually again because their families moved or (very few) because they transferred to less demanding schools (such transfers often took place after students completed what today is called the basic school, which at that time ended with eighth grade — students would then transfer to vocational schools, for example). However, the overwhelming majority of the class remained together from first grade until tenth grade.

Today, while the mobility of the population is somewhat greater, it may be confidently asserted that in an ordinary school the students in a class usually know one another for at least several years. Moreover, although it is now less common for the same teacher to take a class from beginning to end, a teacher will still usually remain with the same class for at least a few years. Naturally, it would not be difficult to point out certain shortcomings of this system, in which the image of the teacher can almost be equated with the image of mathematics — and this is hardly a good thing, particularly if the teacher is not a good teacher. At the same time, certain advantages of this system remain evident: teachers know their classes well, and the classes have time to become accustomed to their teachers' demands; long-term planning in the full sense of the word is feasible, as teachers themselves prepare students for what they will teach in the future. Moreover, such a system in some measure makes the results of a teacher's work more obvious: it would be wrong always to blame the teacher for a poorly prepared class, but at the same time it would definitely be impossible to blame *other teachers* because, in short, there were *no other teachers.*

The required number of students in a class has decreased as the Russian school system has developed. If a class in the 1960s had 35–40 students, now, as a rule, it consists of 25–30 students (here, we are not considering the so-called schools with low numbers of students: a distinctive phenomenon in Russia, where given the existence of tiny villages scattered at great distances from one another it was necessary — and here and there remains necessary — to maintain small schools whose classes could have as few as two or three students).

Elementary school students have the same teacher for all subjects (with the exception of special subjects such as music and art). Teachers of elementary school classes are prepared by special departments at pedagogical institutes and universities as well as special teachers' colleges. The main problem with classes in the teaching of mathematics in elementary school is that not all teachers will have devoted sufficient time to studying mathematics in their past school or college experience, not all teachers regard this subject with interest, and not all teachers have a feeling for its unique character and methodology. The teaching of mathematics can therefore turn into rote learning of techniques, rules, or models for writing down solutions, thereby fostering negative reactions in children between the ages of 7 and 10 and suggesting to them that the main instrument for studying mathematics is memory, not logical reasoning and mental agility. In recent times, the problems associated with mathematics instruction in elementary schools have finally started to receive more active attention from better-prepared experts in mathematics education.

Beginning with fifth grade, mathematics classes are taught by specialist subject teachers who have graduated, as a rule, from the mathematics department either of a pedagogical institute or a university. In today's schools, one also finds former engineers who have lost their jobs for economic reasons and have become re-educated, in some comparatively short program of study, as teachers.

The hours allocated in each class for mathematics consist of the so-called federal — i.e. stipulated by the Ministry of Education — component and other components determined by the region and, to some extent, by the school itself. The number of mathematics classes per week can thus vary both for different years of study and for different schools. Nevertheless, usually in the so-called ordinary class (i.e. a class without advanced study of mathematics and without advanced study in the humanities), 5–6 hours per week are devoted to mathematics. One lesson usually lasts 45 minutes, although in certain periods and in certain schools there have been and continue to be experiments in this respect as well — a 40-minute lesson, a 50-minute lesson, and so on. From seventh grade on, mathematics is split into two subjects: geometry (grades 7–11) and algebra (grades 7–9) or algebra and elementary calculus (grades 10–11).

Students' mathematical preparedness can vary greatly. A diagnostic study conducted by one of the authors of this chapter in two districts of St. Petersburg in 1993 (Karp, 1994) revealed that approximately 40% of tenth graders were unable to complete assignments at the ninth-grade level, while 30% got top grades on such assignments, and approximately 3.5% displayed outstanding results in solving difficult additional problems. (We cite this old study because we believe, for a number of reasons, that its results, at least at the time of the study, accurately reflected the existing state of affairs. At the same time, it must be noted that a very famous school with an advanced course of mathematics was located in one of the districts studied, which naturally would have somewhat improved the average results by comparison with the average level in the whole city.)

The same study revealed a noticeable spread between different schools and classes: in some classes (including classes even outside the aforementioned school), virtually all students received top scores on their assignments, but in other classes none of the students were able to do the work. It is likely that such differences became more profound in subsequent years. At the same time, these differences were not related, as sometimes happens in the United States, for example, to whether the schools were located in the inner city or in the suburbs. Naturally, schools with an advanced course of study in mathematics admit students with a somewhat higher level of preparation. Moreover, in schools with an advanced course of any kind (such as schools with an advanced course in the English language), the average level of mathematics is usually somewhat higher than in ordinary schools. But, not infrequently, ordinary schools with strong teaching and administrative staffs — i.e. schools already having comparatively well-prepared teachers — would go on to become specialized schools with advanced courses of study in various subjects.

In any case, students' levels in, say, a seventh-grade classroom can vary greatly; the same is true even of a tenth-grade classroom (by tenth grade, the most capable students might have already transferred to schools with an advanced course in mathematics and the least interested students would have transferred, for example, to vocational schools). In a class, the teacher sometimes must simultaneously challenge the most gifted students without focusing on them exclusively; select manageable assignments for the weakest students and do as much as possible with them; and work intensively with so-called "average" students, considering their individual differences and selecting the most effective techniques for teaching them.

2.2 *The Mathematics Classroom and Its Layout*

The mathematics classroom, a special classroom in which mathematics classes are conducted, has usually seemed barren and empty to foreign visitors. They see no cabinets filled with manipulatives, no row of computers next to the wall or in the back of the room, no tables nearby piled high with materials of some kind or other. There is no Smartboard and most likely not even an overhead projector.

The large room has three rows of double desks, and each double desk has two chairs before it. The desks are not necessarily bolted down, but even so, no one moves them very much — the students work at their own desks. The front wall is fully mounted with blackboards. Usually, the mathematics teacher asks the school to set up the blackboards in two layers at least on a part of the wall; this would allow the teacher to write on one board and then shift it over to continue writing or to open up a new space with text already prepared for a test or with answers to problems given earlier. Various drawing instruments usually hang beside the blackboards. There may also be blackboards on the side and rear walls of the classroom. Discussing completed assignments on a rear-wall board is not very convenient, because the students must turn around; however, such a blackboard can be reserved for working with a smaller group of students while the rest of the class works on another assignment. The teacher's desk is positioned either in front of the middle row of desks facing the students, or on the side of the classroom against the wall.

Mathematical tables hang on the classroom walls. Usually, these are tables of prime numbers from 2 to 997, tables of squares of natural numbers from 11 to 99, and tables of trigonometric formulas (grades 9–11). The classroom has mounting racks that can be used to display other tables or drawings as needed (such as drawings of sections of polyhedra when studying corresponding topics). Mathematical tables are published by various pedagogical presses, but they may also be prepared by the teachers themselves along with their students. (Recently, paper posters have started getting replaced with computer images which can be displayed on large screens, but for the time being these remain rare.)

On the same racks may be displayed the texts of the students' best reports, sets of Olympiad-style problems for various grades, along with lists of students who first submitted solutions to these problems or with their actual solutions, problems from entrance exams to colleges that students are interested in attending, or problems from the Uniform

State Exam (USE), and so on. A virtually obligatory component of mathematics classroom decoration consists of portraits of great mathematicians. Usually, these include portraits of such scientists as François Viète, Carl Friedrich Gauss, David Hilbert, René Descartes, Sofia Vasilyevna Kovalevskaya, Andrey Nikolaevich Kolmogorov, Gottfried Wilhelm Leibniz, Nikolay Ivanovich Lobachevsky, Mikhail Vasilievich Ostrogradsky, Henri Poincaré, Leonhard Euler, Pafnuty Lvovich Chebyshev, and Pierre Fermat. In class, the teacher might talk about one or another scientist, and draw the students' attention to his or her portrait.

Usually, the classroom features bookcases with special shelves dedicated to displaying models of geometric objects and their configurations. Students might have made these models out of paper. For difficult model-construction projects lasting many hours, students may refer to M. Wenninger's book *Polyhedron Models* (1974); for preparing simpler models, they can rely on the albums of L. I. Zvavich and M. V. Chinkina (2005), *Polyhedra*: *Unfoldings and Problems.* Students having such albums may be given individual or group home assignments to construct a paper model of, say, a polyhedron with certain characteristics and then to describe the properties and features of this polyhedron while demonstrating their model in class. Such student-constructed models may include, for example, the following: a tetrahedron, all of whose faces are congruent scalene triangles; a quadrilateral pyramid, two adjacent faces of which are perpendicular to its base; a quadrilateral pyramid, two nonadjacent faces of which are perpendicular to its base (note that constructing such a model may be difficult but also very interesting for the students); and so on. Any one of these models can be used for more than one lesson of solving problems and investigating mathematical properties. Factory-made models of wood, plastic, rubber, and other materials may also be on display in the classroom. During particular lessons, these models may be demonstrated and studied. Using models for demonstrations differs from using pictures for the same purpose, owing to the higher degree of visual clarity that the former provide, since models can be constructed only if objects really exist, while pictures can even represent objects that do not exist in reality. In contrast to pictures, models allow students not only to see but also to "feel" geometric objects.

Sets of cards for individual questions during class, prepared by teachers over many years, are stored on special shelves in the cabinets. Also kept in the cabinets are notebooks for quizzes and tests. An extremely important part of the classroom may be its library located in the classroom bookcases. In this respect, of course, much depends on the tastes and interests of the teacher (especially since the school usually provides little or no support for creating a library). Meanwhile, the presence of books in the classroom is helpful not only because they may be used during classes or given to students for independent reading at home or for preparing reports, but also because students learn to read and love books about mathematics when teachers talk about, demonstrate, and discuss books.

The library may contain binders of articles from the magazine *Kvant*, books from the popular series "The Little *Kvant* Library," pamphlets from the series "Popular Lectures in Mathematics," and so on. On the other hand, such libraries frequently contain collections of tests and quizzes, educational materials for various grades in algebra and geometry, as well as sets (approximately 15–20 copies) of textbooks and problem books in school mathematics. With multiple copies, students will have the books they need to work at their own desks, while teachers can conduct classes (or parts of classes) including students' work on theoretical materials from one or another textbook or manual or their work on solving problems from one or another problem book. In the past, when teachers had no way of copying the necessary pages, having multiple copies of books was especially important — even now, though it is often more convenient to work with an entire problem book than with a set of copied pages.

Independent classroom work with theoretical materials from the textbook is also extremely important. Helping students develop the skill of working with a book is one of the teacher's goals. Students rarely develop this skill on their own; for this reason, it is desirable for teachers to create conditions in which students will need to call upon this skill, and teachers will be able to demonstrate how to work with a book. For example, a teaching manual containing solutions to various problems, such as V. V. Tkachuk's book *Mathematics for the Prospective College Student* (2006), may be distributed to the students before class, and they may be asked to use it to examine the solution to a problem of medium difficulty involving parametric variables. One can go further and organize a lesson around a discussion on different methods for constructing proofs. Various geometry textbooks may be chosen as materials for this purpose, with students being asked to compare the different techniques employed in them to prove the Pythagorean theorem (grades 8 and 9); to prove that certain conditions are sufficient for a straight line to be perpendicular to a plane (grade 10); or to derive the formula for the volumes of solids of revolution (grade 11). Such lessons are difficult to prepare, but they are extremely informative and useful. However, they are not feasible in all classes, but only in classes with sufficiently interested students.

In sum, we would say that the mathematics classroom has usually had, and indeed continues to have, a spartan appearance not only because Russian schools are poor (although, of course, the lack of funds is of importance: some schools that for one or another reason have more money can have Smartboards, magic markers instead of chalk for writing on the board, and many computers, although this does not necessarily suggest that the computers are being used in a meaningful way). The view is that students should not be distracted by anything extraneous during class. Class time is not a time for leisurely looking around, but for intensive and concentrated work.

3 Certain Issues in Class Instruction Methodology

3.1 *On the History of the Development of Class Instruction Methodology in Russia*

The collection of articles entitled *Methodology of the Lesson*, edited by R. K. Shneider (1935), opens with an article by Skatkin and Shneider (1935) which contrasts the contemporary Soviet lesson with both the type of lesson preceding the Revolution and the one immediately following it and reflecting "left-leaning perversions in methodology." As an example of pre-Revolution schooling, the authors present a lesson about a dog (probably for elementary school students), supposedly taken from teaching guidelines set in 1862. During this lesson, the teacher was supposed to systematically give answers to the following questions: “What is a dog?,” “How big is the dog?,” “What is it covered with?,” “What kind of fur does the dog have?,” and so on. After this, the students themselves were to use the same questions to tell about the dog. The authors conclude: “No mental work is required to master such content: there is nothing to think about here, since there are no relations, connections, causes, explanation” (p. 4)[1]. As for the “left-leaning” lessons, the authors describe a class officially devoted to the poem “The Starling” by the great Russian fable writer Ivan Krylov, during which the teacher launched into a discussion with the students about whether they had ever seen a starling, why starlings are useful, why people build birdhouses, and so on. In this way, the meaning of the fable for the study of Russian language and literature was, in the authors’ opinion, lost.

The “left-leaning” system was criticized for not pursuing the goal of giving the children a “precisely defined range of systematic knowledge.” As an example of arguments directed against knowledge, the authors cite the German pedagogue Wilhelm Lamszus (1881–1965):

> How much of what you and I memorized by rote in mathematics can we really apply in life? All of us, I recall, tirelessly, to the point of fainting, studied fractions, added and subtracted, multiplied and divided proper and improper fractions, all of us diligently converted ordinary fractions into decimals and back again. And now? What has remained of all this? Indeed, what mathematics does a young woman need to know when she becomes a housewife in order to run a household successfully? (p. 6)

Concluding (and largely with reason, it would seem) that such an orientation against knowledge in reality conceals the view that certain portions of the population do not need knowledge, Skatkin and Shneider proceed to formulate a set of requirements for lessons. Among them is the requirement that both the knowledge conveyed and the lessons devoted to it be *systematic*: “A lesson must be organically connected with the lesson before it and prepare the way for the lesson

[1]This and subsequent translations from Russian are by Alexander Karp.

after it" (p. 8). The requirement of *precision* is also emphasized, in relation to both the objectives and the conclusions of each lesson. The *unity of content, of methodological techniques, and of the structure of the lesson as a whole* is put forward as another requirement. In this regard, the authors propose replacing rote learning of the content with conscious and critical acquisition and assimilation.

If Skatkin and Shneider's article was devoted to a theoretical conceptualization of the problem, then L. V. Fedorovich's (1935) article in the same collection gives recommendations (or perhaps even issues orders) about implementing the formulated requirements in practice. Fedorovich writes as follows:

> All of the work must be structured in a way that allows the teacher to pass from the practical problem, the concrete example, to the general law, and after studying the general law with the class, once again to illustrate its application in solving practical problems. (p. 119)

The description of how a lesson must be structured and taught is rigid and precise. For example, the lesson must begin in the following way:

> Everything is prepared for the beginning of the class. The students enter in an organized fashion. All of them know their places (seating is fixed), so there is no needless conversation, above all, no arguments about seats. The students must be taught to prepare their notebooks, books, and other personal materials in 1–2 minutes . . . . The moment when the class is ready is signaled by the teacher, and the students begin to work. (p. 120)

The next recommended step is the checking of homework assignments (the teacher conducts a general discussion and also examines students' notebooks). The teacher must also demonstrate how to complete, and how not to complete, the assignments. All of this should consume 8–12 minutes.

In studying new material, the author recommends:

- Clearly formulating the aim of the lesson for the students;
- Connecting the new lesson with the preceding lesson;
- Identifying the central idea in the new material, paying particular attention to it;

- Viewing the lesson as a link in a unified system and consequently adhering to the common analytic approach;
- Including elements of older material in the presentation of new material;
- Reinforcing the new material;
- Following the textbook in presenting the material.

As for the techniques to be used in presenting the material, the author states that "the techniques must be varied in accordance with the nature of the material itself, the textbooks, and the class's level of preparedness" (p. 124). The author further recommends "mobilizing visual, auditory, and motor perception," using various ways to work with students (verbal communication by the teacher, demonstration, laboratory work, exercises, mental arithmetic, independent work, and so on) and, specifically, using tables and visual aids. Special recommendations are provided on how to avoid mechanical memorization, work on proving theorems, and teach students to construct diagrams (examples show how these should and should not be constructed).

At the end of the class, the teacher summarizes the material, draws conclusions (such as by asking: "What is the theorem that we have examined about?"), and assigns homework. Further, the article indicates that the students are to write down this assignment in their notebooks, tidy up their desks, and leave the classroom in an organized fashion.

To carry out these recommendations, teachers needed to be good at selecting substantive assignments for their students, which was not always the case in practice. At least, the importance of posing substantive questions and recognizing that not everyone was capable of doing so subsequently became a much-discussed topic. For example, an article entitled "Current Survey" (Zaretsky, 1938) published in the newspaper *Uchitel'skaya gazeta* (*Teachers' Newspaper*) contains numerous recommendations about how to pose and how not to pose questions in class:

> Suppose the students have studied the properties of the sides of a triangle. Why not ask them the following: one side of a triangle is 5 cm long, another is 7 cm long; how long might the third side be?

The same article recommends posing questions that are formulated differently from how they are in the textbook: "Thus, in geometry, a student may be asked to give an explanation based on a new diagram." These and other techniques aimed to prevent purely formal memorization of the material. However, judging by the fact that the need to fight against empty formalism in learning remained a subject of discussion for several decades, it was not always possible to implement the recommendations easily and successfully in real life.

On the other hand, the rigidity of the methodological recommendations, even if they were reasonable, could itself cause harm, depriving teachers of flexibility (it should be borne in mind that the implementation of methodological recommendations was often monitored by school administrators who did not always understand the subject in question). As a result, during the 1930s, a rigid schema evolved for the sequence of activities during a lesson: (a) homework review; (b) presentation of new content; (c) content reinforcement; (d) closure and assignment of homework for the next lesson. Going into slightly more detail, we may say that the vast majority of lessons, which always lasted 45 minutes, were constructed in the following manner:

> *Organizational stage* (2–3 minutes). The students rise as the teacher enters the classroom, greeting him or her silently. The teacher says: "Hello, sit down. Open your notebooks. Write down the date and 'class work.' " The teacher opens a special class journal, which lists all classes and all grades given in all subjects, and indicates on his or her own page of the journal which students are absent. On the same page, the teacher writes down the topic of the day's lesson and announces this topic to the students.
>
> *Questioning the students, checking homework, review* (10–15 minutes). Three to five students are called up to the blackboard, usually one after another but sometimes simultaneously, and asked to tell about the material of the previous lesson, show the solutions to various homework problems, talk about material assigned for review, and solve exercises and problems pertaining to material covered in the previous lesson or based on review materials.
>
> *Explanation of new material* (10–15 minutes). The teacher steps up to the blackboard and presents the new topic, sometimes making use of materials from a textbook or problem book in the presentation.

> Until the early 1980s, the same set of mathematics textbooks was used throughout Russia. Sometimes, instead of explaining new material to the students, the teacher asks them to work with a text, and the students read and write an outline of the textbook.
>
> *Reinforcement of new material, problem solving* (10–15 minutes). The students open their problem books and solve the problems assigned by the teacher. Usually, three or four students are called up to the blackboard, one after another.
>
> *Summing up the lesson, homework assignment* (2–3 minutes). The teacher sums up the lesson, reviews the main points of the new material covered, announces students' grades, reveals the topic of the next lesson and the review topic, and assigns homework — which as a rule corresponds to a section from the textbook that covers the new material, sections from the textbook that cover topics for review, and problems from the problem book that correspond to the new material and review topics.

By the 1950s, this schema was already, even officially, regarded as excessively rigid. A lead article in the magazine *Narodnoye obrazovanie* (*People's Education*), praising a teacher for his success in developing in his students a sense of mathematical literacy, logical reasoning skills, and "the ability not simply to solve problems, but consciously to construct arguments," explained the secret behind his accomplishments:

> Boldly abandoning the mandatory four-stage lesson structure whenever necessary, the pedagogue constantly searched for means of activating the learning process. He was "not afraid" to give the students some time for independent work, when this was needed, sometimes even the lesson as a whole, both while explaining new material and while reinforcing their knowledge (Obuchenie, 1959, p. 2).

It is noteworthy, however, that in order to do so, the teacher had to act boldly.

And yet, although the commanding tone of the recommendations cited at the beginning of this section cannot help but give rise to objections, it must be underscored that the problem posed was the problem of constructing an intensive and substantive lesson — a lesson in which the possibility of obtaining a deep education would be offered to all students. This last fact seems especially important. It would be incorrect, of course, to think that Soviet schools successfully taught 100% of their students to prove theorems or even to simplify complicated algebraic formulas — the number of failing students in a class might have been as high as 20%, and far from all students went on to complete the upper grades. Nonetheless, the issue of familiarizing practically all students with challenging mathematics which contained both arguments and proofs was at least considered.

Again, this issue was not always resolved successfully in practice. When the following bit of doggerel appeared in a student newspaper:

> There's no order in the classrooms,
> We can do whatever we please.
> We don't listen to the teacher
> And our heads are in the clouds.

it was immediately made clear that such publications were politically harmful [GK VKP(b), 1953, p. 7]. It may, however, be supposed that discipline in the classroom was indeed not always ideal. Inspectors who visited classes [for example, GK VKP(b), 1947] noted the teachers' lack of preparation and their failure to think through various ways of solving the same problems; the students' inarticulateness and the teachers' inattentiveness to it; and the insufficient difficulty of the problems posed in class and poor time allocation during the lesson.

The reports of the Leningrad City School Board pointed out the following characteristic shortcomings of mathematics classes:

- Lessons are planned incorrectly (time allocation).
- Unacceptably little time is allocated for the presentation of new material.
- The ongoing review of student knowledge is organized in an unsatisfactory fashion — students are rarely and superficially questioned, while homework is checked inattentively and analyzed superficially.
- Systematic review is lacking.
- Work on the theoretical part of the course is weak — conscious assimilation of theory is replaced by mechanical memorization

without adequate comprehension. Teachers are inattentive to students' speech.
- Insufficient use is made of visual aids and practical applications.
- Students' individual peculiarities and gaps in knowledge are poorly studied (LenGorONO, 1952, p. 99).

In other words, practically all of the recommendations cited above met with violations and obstacles. Nonetheless, the unflagging attention to these aspects of the lesson in itself deserves attention.

3.2 *Types of Lessons and Lesson Planning*

The recognition that constructing all lessons in accordance with the same schema is neither always possible nor effective led to the identification of different types of lessons and to the formation of something like a classification of these different types of lessons. Considerable attention has been devoted to this topic in general Russian pedagogy and, more narrowly, in the methodology of mathematics education. Manvelov (2005) finds it useful to identify 19 types of mathematics lessons. Among them — along with the so-called combined lesson, the structure of which is usually quite similar to the four-stage schema described above — are the following:

- The lesson devoted to familiarizing students with new material;
- The lesson aimed at reinforcing what has already been learned;
- The lesson devoted to applying knowledge and skills;
- The lesson devoted to generalizing knowledge and making it more systematic;
- The lesson devoted to testing and correcting knowledge;
- The lecture lesson;
- The practice lesson;
- The discussion lesson;
- The integrated lesson; etc.

As we can see, several different classifying principles are used here simultaneously. The lecture lesson, for example, may also be a lesson devoted to familiarizing students with new material. We will not, however, delve into theoretical difficulties here; they may be unavoidable when one attempts to encompass in a general description all of the possibilities that are encountered in practice. Instead, we will offer examples of the structures of different types of lessons.

A lesson devoted to becoming familiar with new material that deals with "the multiplication of positive and negative numbers," examined by Manvelov (2005, p. 98), has the following structure:

1. Stating the goal of the lesson (2 minutes);
2. Preparations for the study of new material (3 minutes);
3. Becoming acquainted with new material (25 minutes);
4. Initial conceptualization and application of what has been covered (10 minutes);
5. Assigning homework (2 minutes);
6. Summing up the lesson (3 minutes);

For comparison, the practice lesson has the following structure:

1. Stating the topic and the goal of the workshop (2 minutes);
2. Checking homework assignments (3 minutes);
3. Actualizing the students' base knowledge and skills (5 minutes);
4. Giving instructions about completing the workshop's assignments (3 minutes);
5. Completing assignments in groups (25 minutes);
6. Checking and discussing the obtained results (5 minutes);
7. Assigning homework (2 minutes) (Manvelov, 2005, p. 102).

We will not describe the assignments that teachers are supposed to give at each lesson; thus, our description of the lessons will be limited, but the difference between the lessons is nonetheless obvious. Even greater is the difference between them and such innovative types of lessons as the *discussion lesson* or the *simulation exercise lesson*, which we have not yet mentioned and which is constructed precisely as a simulation exercise (as far as we can tell, this type of mathematics lesson is, at least at present, still not very widespread). In contrast to the two lessons described above, in which some similarities to the traditional four-stage lesson can still be detected, the innovative types of lessons altogether differ from any traditional approach.

Naturally, the objectives of a lesson dictate which type of lesson will be taught, and the objectives of the lesson are in turn dictated by the objectives of the teaching topic being covered and by the objectives of the course as a whole. In practice, this means that the teacher prepares a so-called *topic plan* for each course. More precisely, teachers very often do not so much prepare topic plans on their own as adapt the plans proposed by the Ministry of Education. The Ministry proposes a way to divide class hours among the topics of the course, while using one or another Ministry-recommended textbook. Sometimes, teachers use this plan directly; sometimes, they alter the distribution of hours (for example, adding hours to the study of a topic if more hours have been allocated for mathematics at their school than the Ministry had stipulated). In theory, a teacher today has the right to make more serious alterations; but, in practice, the possibilities of rearranging the topics covered in the course are limited — the students already have the textbooks ordered by their school in their hands. Rearranging topics will most likely undermine the logic of the presentation, so the only teachers who dare to make such alterations either are highly qualified and know how to circumvent potential difficulties or are unaware that difficulties may arise. (In fact, district or city mathematics supervisors have the right not to approve plans, but at the present time this right is not always exercised.)

Subsequently, the teacher proceeds to planning individual lessons. Note that it has been a relatively long time since the preparation of a written lesson plan as a formal document was officially required; the plan is now seen as a document for the teacher's personal use in his or her work. At one time, however, a teacher lacking such a document might not have been permitted to teach a class, with all the consequences that such a measure entailed. School administrators frequently demanded that lesson plans be submitted to them and they either officially approved or did not approve them.

Generally speaking, if, say, four hours are allocated for the study of a concept, then the first of these hours will most likely contain more new material than subsequent hours and, therefore, may be considered a lesson devoted to becoming familiar with new material. During the second and third classes, there will probably be more problem-solving, and so those lessons may be considered practice lessons. And the fourth lesson may likely be considered a lesson devoted to testing and correcting knowledge.

Again, however, reality can destroy this theoretical orderliness: new material can (not to say *must*) be studied in the process of solving problems, and therefore it is not always easy to separate becoming familiar with new material from doing a practice on it. The demand that content, methodological techniques, and the structure of the lesson as a whole be unified, as Skatkin and Shneider (1935) insisted, can be fully satisfied only when there is a sufficiently deep understanding of both what the mathematical content of the lesson might look like and how the lesson might be structured (Karp, 2004). In particular, it is necessary to gain a deeper understanding of the role played in class by problem solving and by completing various tasks in general. It is to this question that we now turn.

4 Problem Solving in Mathematics Classes

The methodological recommendations of the 1930s and the subsequent years are full of instructions that the teacher's role must be enlarged. Indeed, teachers were seen as captains of ships, so to speak, responsible for all that occurs in the classroom while at the same time enjoying enormous power there (to be sure, they were endowed with this power as representatives of an even higher power, to which they in turn had to submit, in principle, completely). Teachers were regarded as organizers or, better, designers of lessons, although it would be incorrect automatically to characterize Russian teaching as "teacher-centered" — to use a contemporary expression — especially since this expression usually requires additional clarification. It would be a mistake to equate the dominant role of the teacher as a designer of the lesson, for example, with the lecture style of presentation, or even with a teacher's monopoly of speaking in class. Ideally, the teacher would select and design problems and activities that would enable the students to become aware of new concepts on their own; to proceed gradually and independently from simple to difficult exercises and to further theoretical conceptualization; to think on their own about applying what they have learned; to discover their own mistakes; and so on. This did not rule out that the teacher himself or herself usually posed the questions, summarized the material, or provided the theoretical foundation for various problems.

The mathematics class in the public consciousness was a place where students were taught to think, and this was intended to be achieved through problem solving. In classes devoted to subjects in the natural sciences (physics, chemistry, etc.), the experiment occupies a very important position, and it is precisely in the course of the experiment and the discussion of its organization and results that a student's interests in the subject are formed and developed. In mathematics, then, the equivalent of the experiment is in a sense problem solving. An entire course in mathematics can in fact be constructed — and often is constructed — around the solving of various problems of different degrees of importance and difficulty. Clearly, any theorem may and should be regarded as a problem, and its proof as the solution to that problem. Likewise, the theorem's various consequences should be seen as applications of that problem.

As an example, let us examine one of the most difficult theorems in the course in plane geometry designed by L. S. Atanasyan *et al.* (see, for instance, Atanasyan *et al.*, 2004): the theorem concerning the relations between the areas of triangles with congruent angles. This theorem states that, if an angle in triangle ABC is congruent to an angle in triangle $A_1B_1C_1$, then the arears of the two triangles stand in the same relation to each other as the products of the lengths of the sides adjacent to these angles. In other words, if, for example, angle A is congruent to angle A_1, then $\frac{\mathbf{A}_{ABC}}{\mathbf{A}_{A_1B_1C_1}} = \frac{AB \cdot AC}{A_1B_1 \cdot A_1C_1}$ (**A** is the area). This theorem is very important, since it is then used to prove that certain conditions are sufficient for triangles to be similar, which in turn serves as the basis for introducing trigonometric relations and so forth. In and of itself, too, this theorem makes it immediately possible to solve a number of substantive problems (which will be discussed below). At the same time, its proof is not easy for schoolchildren, and the actual fact that is proven looks somewhat artificial (why should areas be connected with the relations between sides?). The teacher can structure a lesson so that the students themselves ultimately end up proving the required proposition by solving problems that seem natural to them. For example, the teacher may offer the following sequence of problems:

1. Point M lies on the side $\overline{AB}$ of triangle ABC. $\frac{BM}{AB} = \frac{1}{3}$. It is known that the area of triangle ABC is equal to $12\,\text{cm}^2$. What is the area of triangle BMC?
2. Under the conditions of the previous problem, let there be given an additional point K on side $\overline{BC}$, such that $\frac{BK}{CK} = \frac{3}{4}$. What is the area of triangle BMK?
3. Let there be given a triangle ABC and points M and K, on sides $\overline{AB}$ and $\overline{BC}$ of this triangle, respectively, such that $\frac{BM}{AB} = \frac{3}{7}$ and $\frac{BK}{CK} = \frac{2}{9}$. It is known that the area of triangle ABC is equal to $\mathbf{A}$. Find the area of triangle BMK.
4. Given a triangle ABC, let M be a point on the straight line $\overleftrightarrow{AB}$ such that A lies between M and B, and such that $\frac{BM}{AB} = \frac{9}{5}$. Let K be a point on side $\overline{BC}$, such that $\frac{BK}{CK} = \frac{4}{7}$. It is known that the area of triangle ABC is equal to $\mathbf{A}$. Find the area of triangle BMK.

The first of these problems is essentially a review — the students by this time have usually already discussed the fact that, for example, a median divides a triangle into two triangles of equal area, since the heights of the two obtained triangles are the same as the height of the original triangle, while their bases are twice as small. Consequently, in the problem posed above, it is not difficult to find that the area of the obtained triangle is three times smaller than the area of the given triangle. The second problem is analogous in principle, but involves a new step — the argument just made must be applied for a second time to the new triangle. The third problem combines what was done in the first and second problems, but now the students must themselves break the problem down into separate parts, i.e. to make an additional construction. Moreover, the numbers given are somewhat more complicated than the numbers in the preceding problems. The fourth problem is identical to the third in every respect except that the positions of the points A, B, and M are somewhat different — in other words, the diagram will have a somewhat different appearance [Figs. 1(a) and 1(b)].

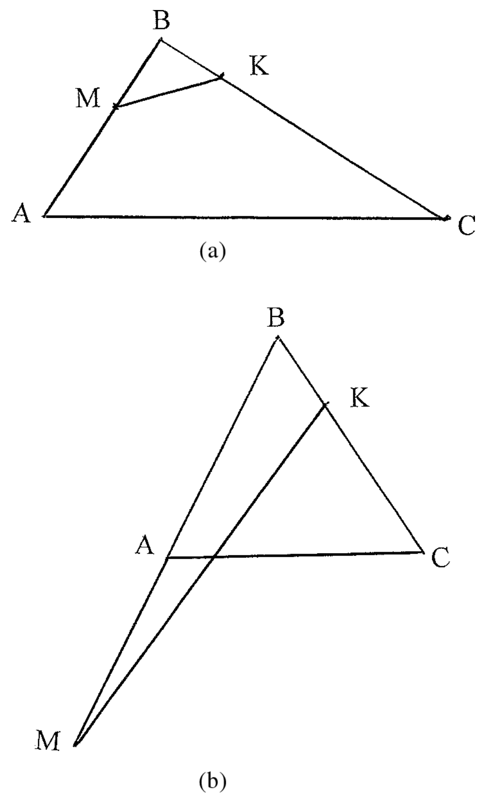

Fig. 1.

In this way, the whole idea of the theorem's proof is discussed. What is required to complete the proof of the theorem? It is still necessary to make a transition from expressing the idea in terms of numerical values to expressing it in terms of general relations. The expression "find the area of the obtained triangle, based on your knowledge of the area of the given triangle" must be replaced with an expression about relations between areas (which will be natural, since it is already clear why this relation is needed). Finally, it is necessary to examine the general case, where two different triangles with congruent angles are given, rather than two triangles with a common angle, i.e. it must be shown that the general case can be reduced to the case that has been investigated, by "superimposing" one triangle on the other. All of this can usually be done by the students themselves, i.e. they can be told to carry out the proof of the theorem as a final problem. But even if teachers decide that it would be better if they themselves sum up the discussion and draw the necessary conclusions, the students will be prepared.

It must be pointed out here that genuine problem solving is often too categorically contrasted with the solving of routine exercises. The implication thus made is that in order to involve students in authentic problem solving in class, they must be presented with a situation that is altogether unfamiliar to them. Furthermore, because it is in reality clear to everyone that nothing good can come of such an exercise in the classroom, students are in fact not given difficult and unfamiliar problems. Instead, they receive either mere rhetoric or else *long* problems or *word* problems in place of *substantive* problems.

The whole difference between solving problems in class and solving problems chosen at random at home lies in the fact that in class the teacher can help — not by giving direct hints, but by organizing the problem set in a meaningful way. Indeed, even problems that seem absolutely analogous (such as problems 1 and 2 above) in reality demand a certain degree of creativity and cannot be considered to be based entirely on memory; this has been discussed, for example, by the Russian psychologist Kalmykova (1981). A *structured system of problems* enables students to solve problems that are challenging in the full sense of the word. Yes, the teacher helps them by breaking down a difficult problem into problems they are capable of solving, but precisely as a result of this the students themselves learn that problems may be broken down in this way and thus become capable of similarly breaking down problems on their own in the future. This is precisely the kind of scaffolding which enables students to accomplish what they cannot yet do on their own, as described by Vygotsky (1986).

It is important to emphasize that the program in mathematics has been constructed and remains constructed (even now, despite reductions in the amount of time allocated for mathematics and increases in the quantity of material studied) in such a way that it leaves class time not only for introducing one or another concept, but also for working with it. Consequently, even in lessons which would be classified as lessons devoted to reinforcing what has already been learned (according to the classification system discussed above), students not only review what they have learned, but also discover new sides of this material. To illustrate, let us briefly describe a seventh-grade lesson on "Polynomials," which follows a section on the formulas for the squares of the sums and differences of expressions.

> At the beginning of the lesson, the teacher conducts a "dictation:" she dictates several expressions, such as "the square of the sum of the number *a* and twice the number *b*" or "the square of the difference of three times the number *c* and half of the number *d*." The class, as well as two students called up to the blackboards, write down the corresponding algebraic expressions and, manipulating them in accordance with the formulas, put them into standard form. The blackboards are positioned in such a way that the work of the students at the blackboards cannot be seen by the rest of the class. Once they complete the dictation, students in neighboring seats switch notebooks, the class turns to face the blackboards, and all the students together check the results, discussing any mistakes that have been made (students in neighboring seats check one another's work).
>
> Then the class is given several oral problems in a row, which have also been written down on the blackboard, and which require the students to carry out computations. Without writing anything down, the students determine each answer in their minds and raise their hands. When enough hands are raised, the teacher asks several students to give the answer and explain how it was obtained. The problems given include the following:
>
> 1. $21^2 + 2 \cdot 21 \cdot 9 + 9^2$
> 2. $2009^2 + 2010^2 - 4020 \cdot 2009$
> 3. $(100 + 350)^2 - 100^2 - 350^2$
> 4. $\frac{17^2 + 2 \cdot 17 \cdot 13 + 13^2}{900}$
> 5. $\frac{32^2 - 2 \cdot 32 \cdot 12 + 12^2}{13^2 + 2 \cdot 13 \cdot 7 + 49}$
>
> In a final problem, the teacher deliberately writes down one number illegibly (it is denoted as $\otimes$): $\frac{50^2 - \otimes + 30^2}{13^2 + 2 \cdot 13 \cdot 7 + 49}$. The students are then asked what number should be written down in order to make this expression analogous to the previous one.
>
> After solving and discussing these problems, the students are asked to solve several problems involving simplifications and transformations. The students work in their notebooks. In conclusion, students

are called up to the blackboards to write down the answers to these problems, one by one, along with necessary explanations. The problems given include the following:

1. Write each of the following expressions in the form of a square of a binomial, if possible: (a) $x^2 + 16 - 8x$; (b) $4t^2 + 12t + 9$.
2. Find a number k such that the following expression becomes the square of a binomial: $z^2 + 8z + k$.
3. Simplify the following expressions: (a) $a^2 - 2a + 1 - (a + 1)^2$; (b) $2m^2 - 12m + 18 - (3 - m)^2$; (c) $(m - 8)^2 - (m - 10)(m - 6)$; (d) $\frac{(x+2)^2+4(x+2)+4}{(x+4)^2}$.

Subsequently, the teacher inquires about deriving the formula for the square of the sum of a trinomial and asks the students to discuss the following, allegedly correct formula:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 3ac + 4bc.$$

After the students discuss this formula, they are asked to derive the correct formula on their own (the result is written down on the blackboard).

The lesson concludes with the students being asked to prove that, for any natural values of n, the expression $9n^2 - (3n - 2)^2$ is divisible by 4 (more precisely, this problem is given to those students who have already completed the previous problem).

As we can see, it would be somewhat naive to attempt to describe this lesson without taking into account the specific problems that were given to the students. Collective work alternates with individual work here, and written work alternates with oral work. The teacher, even when using the rather limited amount of material available to seventh graders, tries to teach them not a formula, but the subject itself. For this reason, connections are constantly made with various areas of mathematics and various methods of mathematics — the students communicate mathematically, make computations, carry out proofs, evaluate, check the justifiability of a hypothesis, and construct a problem on their own (even if relying on a model). They apply what they have learned, both while carrying out computations and, for example, while proving the last proposition concerning divisibility, but they also derive new facts (such as a new formula).

On the one hand, nearly all of the problems are different; no problems are different only by virtue of using different numbers and are otherwise identical. On the other hand, the problems given to the students echo one another and, to some extent, build on one another. For example, in the computational exercise No. 1, the formula is applied in standard form, while in No. 2 a certain rearrangement must be made. Problems involving the simplification of algebraic expressions recall the computational problems given earlier. The problem in which students are asked to determine a k to obtain the square of a binomial has something in common with the problem containing the illegible number, and so on — not to mention that the formulas repeated during the first stage become the foundation for all that follows.

The lesson is structured rather rigidly in the sense indicated above, i.e. in terms of the presence of links and connections that make the order of the problems far from arbitrary. At the same time, a lesson with such content requires considerable flexibility and openness on the teacher's part. For example, the hypothesis that $(a + b + c)^2 = a^2+b^2+c^2+2ab+3ac+4bc$ may be rejected by the students for various reasons — say, because the expression proposed is not symmetric (the students will most likely express this thought in their own way, and the teacher will have to work to clarify it), or simply because when certain numbers are substituted for the variables, e.g. $a = b = c = 1$, the two sides of the equation are not equal. However, the students might also express opinions that they cannot convincingly justify (for example, that the coefficients cannot be 3 and 4 because the formulas studied previously did not contain these coefficients). The teacher must have the ability both to get to the bottom of what students are trying to say in often unclear ways, and to take a proposition and quickly show its author and the whole class that it is open to question and has not been proven.

One of the authors of this chapter (Karp, 2004) has already written elsewhere about the complex interaction between the mathematical content and the pedagogical form of a lesson. Sometimes the teacher is able to achieve an interaction between content and form that has an emotional effect on the students comparable to the effect made by works of art. However, even given the seemingly simple composition of the lesson examined above, the choice of adequate pedagogical techniques for the lesson is essential. It is difficult for students in general and for seventh graders in particular to remain at the same level of concentration for the entire 45 minutes (without suggesting that issues of discipline can always be resolved and only through successful lesson construction, we will nonetheless say that it would be ill-advised to expect 13-year-old children to sit quietly and silently during a lesson in which they have nothing to do or, on the contrary, are given assignments that are too difficult for them). Consequently, questions arise about how more and less intensive parts of a lesson can alternate with one another, and about the rhythm and tempo of the lesson in general. In the lesson examined above, a period of intense concentration (dictation) was followed by a less intense period, during which the students' work was checked; intense oral work was followed by more peaceful written work. Consequently, collective work was followed by individual work, with students working at their own individual speeds. At this time, the teacher could adopt a more differentiating approach, perhaps even giving some students problems different from those being solved by the whole class. An experienced teacher almost automatically identifies such periods of differing intensities during a lesson and selects problems accordingly.

Note that group work, which has become more popular in recent years partly because of the influence of Western methodology, is still (as far as can be judged) rarely employed; this contrasts with working in pairs, including checking answers in pairs, as exemplified in the lesson examined earlier. Without entering into a discussion on the advantages and disadvantages of working in groups, and without examining the difficulties connected with frequently employing this approach, we should say that this approach has not been traditionally used in Russia (as we noted, even the classroom desks are arranged in such a way that it is difficult to organize group work). On the other hand, administrative fiat in Russia and the USSR has imposed so many methodologies which were declared to be the only right methodologies that Russian teachers usually react skeptically to methodologies that are too vehemently promoted. The creation of a problem book for group work presents an interesting methodological problem, i.e. the creation of a collection of substantive problems in school materials for the solving of which group effort would be genuinely useful, so that working in groups would not simply involve students comparing and coordinating answers or strong students giving solutions to weaker ones. As far as we know, no such problem book has yet been published in Russia.

It should not be supposed, of course, that every lesson must be constructed as a complicated alternation of various pedagogical and methodological techniques. Mathematics studies in general and mathematics lessons in particular can to some degree consist of monotonous independent work involving the systematic solving of problems. The difference between this kind of work and completely independent work on problems chosen "at random" lies in the fact that the problems in the former case are selected according to some thematic principle or because the solutions involve the same technique and so enable the students to better grasp the material. As an example, let us examine part of a problem set from a course in geometry for a class that is continuing to study relations between the areas of triangles with congruent angles, which we mentioned earlier:

1. Points M and N lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, of triangle ABC. $\frac{AM}{BM} = \frac{5}{3}$; $\frac{BN}{BC} = \frac{7}{8}$. Find: (a) the ratio of the area of triangle BMN to the area of triangle ABC; (b) the ratio of area of quadrilateral $AMNC$ to the area of triangle BMN.
2. Triangle ABC is given. Point A divides segment $\overline{BK}$ into two parts such that the ratio of the length of BA to that of AK is 3:2. Point F divides segment BC into two parts such that the ratio of their lengths is 5:3. The area of triangle BKF is equal to 2. Find the area of triangle ABC.
3. The vertices of triangle MNK lie on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, of triangle ABC in such a way that $AM{:}MB = 3{:}2$; $BN = 6NC$; and K is the midpoint of AC. Find the area of triangle MNK if the area of triangle ABC is equal to 70.
4. $ABCD$ is a parallelogram. Point F lies on side $\overline{BC}$ in such a way that $BF{:}FC = 5{:}2$. Point Q lies on side $\overline{AB}$ in such a way that $AQ = 1.4QB$. Find the ratio of the area of parallelogram $ABCD$ to the area of triangle DFQ.

As we can see, the set begins with a problem that the students already know. They can now solve it by directly applying the theorem to two triangles that have the common angle B. In problems 1(b) and 2, the application of the theorem becomes somewhat less straightforward — in problem 1(b), the students have to see that the area of the quadrilateral AMNC and the area of the triangle BMN together make up the area of the triangle ABC, while in problem 2 they must find the area of the given triangle ABC rather than of the obtained triangle, as was the case earlier. In problem 3, the basic idea has to be applied several times. In problem 4, this must be done in a parallelogram, which must additionally be broken down into triangles. Such a problem set can be expanded with more difficult problems.

Note that such problems may be used in another class: with eleventh graders when reviewing plane geometry. In that case, it would be natural to continue the series using analogous problems connected with the volumes of tetrahedra and based on the following proposition:

> If a trihedral angle in tetrahedron $ABCD$ is congruent to a trihedral angle in tetrahedron $A_1B_1C_1D_1$ then the volumes of the two tetrahedra stand in the same ratio to each other as the lengths of the sides that form this angle. For example, let the trihedral angle $ABCD$ be congruent to the trihedral angle $A_1B_1C_1D_1$. Then $\frac{\mathbf{V}_{ABCD}}{\mathbf{V}_{A_1B_1C_1D_1}} = \frac{AB \cdot AC \cdot AD}{A_1B_1 \cdot A_1C_1 \cdot A_1D_1}$ (where **V** represents volume).

Until now, we have emphasized the importance of problem sets. But sometimes it makes sense to construct a lesson around a single problem. For example, the outstanding St. Petersburg teacher A. R. Maizelis (2007) would ask his class to find as many solutions as they could to the following problem:

> Given an angle ABC and a point M inside it, draw a segment $\overline{CD}$ such that its endpoints are on the sides of the angle and the point M is its midpoint.

Students would offer many different solutions (usually in one way or another involving the construction of a parallelogram whose diagonals intersected at the point M). After this, the same kind of problem was posed, not about an angle and a point M but, say, about a straight line, a circle, and a point M. Usually, none of the previously offered solutions could be transferred to this new problem, which nevertheless became easy to solve once the students realized that they had to make use of the properties of central symmetry. Such a lesson expanded their understanding of the meaning of the theorems they had learned earlier; it helped them evaluate the possibilities of applying transformations; and, more broadly, it trained them to become genuine problem solvers who discovered aesthetic pleasure from the actual process of solving problems.

5 Epilogue: Bad Lessons, and What One Would Like to Hope for

Above, we spoke mainly about "good" lessons. It would be misleading, of course, to claim that all lessons in Russia could be so characterized. Paraphrasing Leo Tolstoy's famous line about unhappy families, one could say that every bad lesson is bad in its own way. The system of rigid monitoring and uniform rigid requirements is a thing of the past. Searching for a general formula for failure, and thus for a general prescription for turning bad lessons into good ones, is futile. Yet, certain patterns can still be identified.

The system of intensive work and high demands in class, described above, presupposed systematic work outside of class as well. In the 1930s, and indeed much later also, teachers were explicitly required, in addition to normal classroom lessons, to conduct additional lessons with weak students. These lessons, in turn, were not always successful; often enough, they consisted of "squeezing out" a positive grade. Yet their very existence (even as a form of punishment for students who had failed to fulfill what was required of them in time and were for this reason forced to spend time after school) played a definite role. Nor did it occur to anyone to pay teachers extra wages for such classes — these were considered a part of ordinary work. On the other hand, from a certain point on, a well-developed system for working with the strongest students existed. In addition to the fact that the strongest students would leave ordinary schools to attend schools with an advanced course in mathematics, there existed mathematics clubs or "math circles" (including clubs in schools), optional classes, regularly assigned difficult problems, and the like, which to some extent facilitated the productive engagement of the strongest students as well.

In using the past tense, we do not wish to imply that working with students outside class boundaries has now receded into the past. This form of work still exists, although — probably above all for economic reasons — not everywhere. Meanwhile, unoccupied students and students who have not been given enough to do invariably create problems during the lesson. To repeat, this is not the only cause of poor discipline in some schools, which affects all classes, particularly mathematics classes. While it is impossible to resolve social problems by relying exclusively on a teacher's skills, the absence of such skills may exacerbate such problems and give rise to discipline problems where no deep social reasons for them exist.

As one negative development of recent times, we should mention a specific change in the attitude of some teachers. It is fitting to criticize the perfunctory optimism of Soviet era schools, with their cheerful slogans such as: "If you can't do it, we'll teach you how to do it, and if you don't want to do it, we'll force you to do it!" Nevertheless, the system as a whole encouraged teachers to believe that practically all students must be raised to a certain level (even if this was not always possible in practice). Everyone recognized the importance of mathematics education in this context. Posters with the words of the great Russian scientist Mikhail Lomonosov were hung (and have hung to this day) in virtually every mathematics classroom: "Mathematics must be studied if for no other reason than because it sets the mind in order." Who would argue that the mind should not be set in order? Again, the authors of this chapter would like to believe that society as a whole has largely preserved its respect for the study of mathematics and that this gives reason to hope that current difficulties will be overcome. Yet in all fairness it must also be pointed out that the justifiable fight against a fixation on universal academic advancement has sometimes turned into an unwillingness to try to teach students (we should qualify this statement at once, by saying that it is based on observations, not on systematic studies or statistics — we have no such data at our disposal).

In classes, this is evidenced by the fact that even those teachers who adhere to the form of the intensive lesson are not concerned about its results. Probably the worst class ever observed by one of the authors of this chapter was a class that he visited once while supervising a school for juvenile delinquents. The problem had nothing to do with discipline, as might have been expected. On the contrary, the discipline was excellent, and the teacher began by energetically conducting a mathematical dictation; he then explained new material, making use of a variety of techniques, and this was followed by independent work and a mathematical game — and so on and so forth. The trouble with this display of pedagogical and methodological fireworks was that the material being studied was eighth-grade material, while all of the students — as was obvious from their answers — barely knew mathematics at a fourth-grade level. A strange exercise was taking place during which no one learned anything. The teacher, however, was not in the least disconcerted by the students' absurd answers — the class, as it were, had a legitimate right to be considered weak.

This example is to some extent exceptional, but the absence of a goal truly to teach students and the willingness to ignore reality may be the most important reasons for bad classes, i.e. classes that fail to teach students, in ordinary schools. Indeed, its manifestations may be observed in selective schools as well, when teachers set goals they know are unrealizable, such as attempting to cram into a single lesson material that would be challenging to cover in three lessons — since, after all, the children are good students. However paradoxical it may seem, Russian respect for mathematics sometimes has negative consequences in such cases: both the children and their parents make the mistake of thinking that a large quantity of work implies a high quality of education.

The art of being a teacher in any country, including Russia, presupposes the ability to choose problems and to leave enough time for their solution, to determine what will be tiring for the students and what will give them a chance to rest. It presupposes the ability to know a large number of useful sources and to pose the right questions on the spot in the classroom, displaying flexibility and departing from what was previously planned. And the list goes on. It is not difficult to provide examples of Russian classes in which the teachers themselves did not really know the subject and thus could not teach their students, or in which the predetermined lesson plan collapsed because the very first activity consumed all of the class time, making the activity pointless. Yet, a teacher's inability to plan adequate time for a lesson and even insufficient knowledge of the subject may usually be overcome through systematic and persistent work — and, above all, through a commitment to overcoming these weaknesses.

The traditions of Russian mathematics education, including those of conducting and constructing lessons, took shape as part of the complex and often frightening development of Russian history. People sometimes became teachers of mathematics who, under different circumstances, might have been department chairs at leading universities. The rigid and merciless system forced teachers to work long and hard, usually without minimally adequate compensation. The system raised mathematics to a privileged position, while often at the same time destroying existing scholarly traditions and instruction of the humanities. This same system gave rise to a meaningless formalism in the teaching of mathematics and to a fear of deviating from approved templates.

Nevertheless, over the course of a complicated development in a country that possesses enormous human and cultural resources, traditions of intensive, genuine, and fully instructive mathematics education emerged. Regardless of the circumstances that brought these wonderful teachers to schools, these individuals created models which all teachers to this day can aspire to match. These are models of an attitude toward one's work and its inherent problems, models of relations with students, models of lessons taught. These models do not concern the details, which inevitably change and are renewed over time, as the authors of this chapter witnessed when certain topics that had previously been deemed important were dropped from the curriculum; even less do they concern technological implements, such as the slide rule. At stake, rather, are models of how to achieve the goal of genuinely teaching and developing children during every class, and models of how to employ a rich palette of techniques, methods, and problems for doing so. These models continue to exert their influence — they have been seen by thousands of people, including those who became teachers and those who became parents, and who want their children to have education similar to what they once had. It is important to remember that these models have to a certain extent been reflected in textbooks and problem books, which in their turn oriented and educated new teachers. It is on the vitality of existing traditions that one would like to pin one's hopes.

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2

The History and the Present State of Elementary Mathematical Education in Russia

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1 Introduction

The final decade of the 20th century and the first decade of the 21st century witnessed a worldwide effort to advance the state of mathematical education. Not surprisingly, UNESCO had named the year 2000 as World Mathematical Year. In its push toward better mathematical education, Russia, alongside other nations, must strive to balance innovation and valuable tradition that has stood the test of time. In this chapter, we will examine the history of elementary mathematical education in Russia and consider its present state as well as the prospects of its development in light of the recently introduced "second-generation" Federal Educational Standard of 2010 (http://standart.edu.ru/catalog.aspx? CatalogId=531).

2 The History of Arithmetical Education in Russia During the 10th–18th Centuries

The origins of Russian mathematics can be traced to the 10th–12th centuries (Kolyagin, 2001). The so-called "Russian Justice" — a legal code compiled in the 11th and 12th centuries which has survived until today — contains certain curious arithmetical problems alongside the more typical prescriptions. One of the first Russian mathematicians was Kirik of Novgorod (Kolyagin, 2001, p. 11; Polyakova, 1997, p. 23), who had produced a mathematical treatise as early as 1136. Kirik performed his calculations with an abacus and used wax tablets for scratch paper. Using the lunar and solar cycles, he was able to calculate time, the shifting date of Easter, the leap year, and so on. He made use of fractions when describing the precise time of day. Kirik's handbook was used in the so-called elementary grammar schools during the times of Yaroslav the Wise and enjoyed widespread influence. Tragically, the cultural development of Kievan Rus was cut short by the Tatar-Mongolian invasion.

In the 16th and 17th centuries, mathematics was considered a practical skill associated with housekeeping and trade, and was not made part of elementary education. It was transferred orally and practically (with the use of the abacus). The first handbooks (as opposed to textbooks) appeared around this time: *Cipher-Counting Science and Convenient Counting*, among others (Kolyagin, 2001; Polyakova, 1997).

The renowned scholar A. I. Sobolevsky (1857–1929) believed that the large number of manuscripts that have survived, despite the great fires of the 15th–17th centuries, suggest that these texts were copied by thousands of scribes and intended for wide readership (Kolyagin, 2001, p. 11). Arithmetical manuscripts (studied by the historian of mathematics V. V. Bobynin) typically had a foreword that located arithmetic among the seven "liberal arts": grammar, rhetoric, dialectic, music, arithmetic, geometry, and astronomy. Together, these "arts" constituted the core of higher learning in the medieval age. A typical manuscript included the numeral system, the four arithmetical operations with natural numbers, calculation, fractions, and so forth. The texts provided the reader with arithmetical rules, extensively illustrated with various exercises ranging from simple to complex. There were problems involving proportional division of property, estimating the need for containers, mixtures, payments to business associates and clerks, division of profit, interest, and other topics. Here is an example of a problem from such a manuscript (Kolyagin, 2001, p. 16):

> How many days will it take for the wife to drink a keg of kvas if she and her husband together drink a keg in 10 days, while the husband alone can drink it in 14 days?
> *Solution.* Take 10 from 14: there remains 4. Say, 4 gives 10. What will give 14? Multiply 14 by 10, and get 140; divide 140 by 4, and get 35 days. It will take 35 days for the wife to drink a keg of kvas by herself.

The first printed work in mathematics, *The Book of Convenient Counting*, which comprised multiplication tables up to 100 · 100 and was written in Slavonic numeration, was published in Moscow in 1682. The first Russian textbook proper was *Arithmetic* (Magnitsky, 1703), written by the remarkable Russian mathematician and pedagogue, L. F. Magnitsky (1669–1739), in two volumes (over 600 pages). The section of the book dealing with arithmetic proper includes the Arabic numeral system, tables of addition and multiplication for positive integers (demonstrating interchangeability of operations), operations with whole numbers, currency and measuring systems of various countries, fractions, proportions, progressions, square and cube roots, and problems in applied geometry. A great deal of attention is given to general discussions on mathematics. Magnitsky notes: "Arithmetic, or numeration, is an honest art, envy-free, readily grasped by all, and wholly useful … ." The material is presented in question-and-answer form. Each new mathematical rule is preceded by a simple example, followed by a general formulation of the rule and several analyzed problems, mostly of a practical nature (Kolyagin, 2001; Polyakova, 1997). The book contains numerous illustrations and borrows much of its terminology and content from its manuscript predecessors. Here are a few typical problems from Magnitsky's textbook:

- A man was selling a horse for 156 rubles. The buyer said that the price was too high. The seller then proposed: "Buy only the nails in the horseshoes. And take the horse gratis. There are six nails in every horseshoe. Pay a quarter-copeck for the first nail, a half-copeck for the second nail, a whole copeck for the third nail, and so on for the full set." The buyer agreed. Find out what price the buyer ended up paying. (Magnitsky, 1703, p. 185)

- A man is sent from Moscow to Vologda and ordered to travel 40 *verst* each day. The following day another man is sent along the same route and ordered to travel 45 *verst* each day. In how many days will the second man overtake the first? (Magnitsky, 1703, p. 218)

Magnitsky's *Arithmetic* is a unique work. For over half a century, it was both a textbook and an encyclopedia of mathematical knowledge. M. V. Lomonosov referred to it as "the gateway to my education." The historian of mathematics V. V. Bobynin believed that "in all of Russian scientific and mathematical literature one may scarcely find a book of historical significance comparable to Magnitsky's *Arithmetic*" (Kolyagin, 2001, p. 20).

Magnitsky's textbook was used in the School for the Mathematical and Navigational Sciences, founded in Moscow in 1701 by a decree of Peter the Great. Magnitsky served as one of its instructors (from 1701 until his death), alongside specially retained British pedagogues. Beginning in 1714, the school graduated not only seamen, engineers, civil servants, and others, but also teachers of elementary "arithmetic" schools, funded by the government, which had by that time appeared in several major cities. The school's curriculum included arithmetic and geometry, among other subjects.

The reign of Peter I is traditionally recognized as the beginning of Russian mathematics and teaching methodology (Kolyagin, 2001). Among textbooks of this period we find *A Manual of Arithmetic to Be Used in the School of the Imperial Academy of Sciences* (1735), by L. Euler[1] (1707–1783). It was later to serve as the basis for the wonderful textbooks of N. G. Kurganov (1725–1796): *Universal Arithmetic* (1757) and *Numerary* (1771). These texts set out a systematic course of mathematics using accessible language and offering

[1] Leonard Euler worked at the Academy of Sciences in St. Petersburg from 1727 through 1741, and again from 1766 through 1783. He is buried in St. Petersburg.

logical explanations and illustrations with challenging problems and exercises.

M. V. Lomonosov had developed the "Rule of Moscow Gymnasia" (1775), which devoted a special place to mathematics. The first year was devoted to the study of arithmetic, followed by applied geometry, trigonometry, and plane geometry in the second year. Lomonosov proposed a method of using systematic mandatory exercises (in class and at home) as well as optional assignments for homework (Kolyagin, 2001).

Under the "Charter for Public Schools of the Russian Empire" (1786), arithmetic was included among the subjects covered in the "first stage" (first and second grades) and mathematics among the subjects of the "second stage" (third and fourth grades). A school-day system with class periods was introduced at this time: the teacher presented a lesson to the entire class; the students proceeded to solve a variety of problems typically pertaining to daily activities (Kolyagin, 2001). Practical application remained the primary focus of mathematical education until the end of the 18th century: students were generally taught skills that had practical value in their daily lives.

3 Elementary Mathematical Education in Russia in the 19th and Early 20th Centuries (through 1917)

The methodology for teaching arithmetic took shape in the 19th century. This period was marked by debates between two different approaches to teaching arithmetical operations with whole positive numbers at the primary level. According to the first approach, native to Russia, students learned numbers (derived through counting) and the decimal numeration system, followed by arithmetical operations and calculation techniques. Among the proponents of this approach were P. S. Guriev (first manual 1832), A. I. Goldenberg, V. A. Latyshev, and others. The second approach, based evidently on foreign models, was to study numbers through the so-called monographical method (A. Grube, V. A. Evtushevsky, and others).

3.1 *The Method of Learning Operations*

The founder of the teaching methodology for arithmetic in Russia was P. S. Guriev[2] (1807–1884) (Andronov, 1967; Lankov, 1951). Guriev argued for a concentric arrangement of subjects and identified three "circles": the numbers 1–10, the numbers up to 100, and all other numbers. He believed that study must proceed from the concrete to the abstract and advocated what is today called "developmental education." He paid special attention to independent work:

> The crucial task is to foster in your pupil a sense of independence, to reveal to him the brightest, the most luminous aspect of learning, so that he may always thirst after knowledge, and experience even in the narrow sphere of his present studies joy and satisfaction in the discovery of any new knowledge, any new truth (Lankov, 1951, p. 31).

To guide the independent studies of his pupils, Guriev devised his first teaching materials: "Arithmetical sheets, gradually arranged from the simplest to the most difficult" (Guriev, 1832). These sheets contained exercises, problems, and rules for performing arithmetical operations. After explaining the materials, the teacher could hand out the cards in accordance with individual students' abilities. In this manner, Guriev laid the foundation for the differentiated approach to independent student work in arithmetic.

Unfortunately, Guriev did not write textbooks: his ideas never gained wide currency and were never able to compete with the "monographical method of learning the numbers." D. D. Galanin wrote of P. S. Guriev as follows:

> It is indisputable that Russian pedagogy had a far better understanding of mathematical education than the German teacher of the times, and we can only regret the fact that the subsequent shift in society stifled the tender shoots of this sound pedagogical trend, and that

[2] P. S. Guriev was the son of the academician S. E. Guriev and a student of the academician P. N. Fuss, grandson of L. Euler. He served as a teacher for 30 years, then as superintendent of the Gatchina Institute for the Orphaned (training teachers of regional academies), a teacher at a school for orphans (which he financed out of his own pocket), a trustee of various county schools, and editor of the journal *Russian Pedagogical Courier* (1857–1861).

> a fad for German methodology placed the school system on a false psychological foundation (Galanin, 1915, p. 217).

3.2 *The Monographical Method of Learning the Numbers*

This method was devised by the German methodologist A. Grube[3] (1816–1884) and is based on the idea of I. G. Pestalozzi (1746–1827) that placed visualization at "the basis of all knowledge." Grube's system gave primacy of place to the "principle of a comprehensive study of numbers," with a "contemplation of number." Each number was related to and measured against its predecessors by means of subtraction or division. Students were not taught the decimal numeration system, arithmetical operations, or applications of arithmetic in everyday life. Lankov (1951) wrote:

> The study of arithmetic according to Grube's method is tedious and "dulls the wits" of the students. Having learned several numbers, they come to expect nothing other than the same sad prospect of endless combinations without any pause for reflection upon material covered. The sense of eternal monotony weakens the students' resolve and destroys their interest. (p. 50)

V. A. Evtushevsky[4] (1838–1888) adapted Grube's method for Russian schools: children studied in detail the numbers 1 through 20, as well as those numbers under 100 that have a few prime divisors (24, 30, 32, etc.). Numbers over 100 were studied later in relation to arithmetical operations. The simplicity of Evtushevsky's approach (from the teacher's perspective) made it generally popular,[5] although this approach was later criticized.

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[3] Grube's *Guidebook for Counting in Elementary Schools, Based on the Heuristic Method* (first German edition in 1842) was published in G. F. Ewald's Russian translation in 1873.

[4] V. A. Evtushevsky, *Exercise Book in Arithmetic* (1872) and *Methodology of Arithmetic* (1872).

[5] D. L. Volkovksy (1869–1934) attempted to resurrect this method for teaching numbers in Russia in his book *A Child's World in Numbers* (1913–1916).

At the same time, Evtushevsky was one of the first to emphasize the developmental and formative significance of mathematical education. He saw its developmental power in the study of the theory and mechanisms of calculation, and in the application of theoretical knowledge to practical exercises. The mechanism of calculation is "a language, by means of which mathematics expresses its ideas, poses and answers its questions" (Evtushevsky, 1872, p. 24). The application of this language and theoretical foundations to practical problems was, according to Evtushevsky, the most significant instance of the pedagogical effect that the study of mathematics had upon the development of students' cognitive skills. Unfortunately, it appears that the majority of Evtushevsky's general principles were not realized in his handbooks, where he had applied his talents to improving a fundamentally flawed "method of learning the numbers."

The battle against this formal method was waged for some time by many Russian educators (A. I. Goldenberg, V. A. Latyshev) and other members of various intellectual circles.[6] A. I. Goldenberg (1837–1902) had made a decisive contribution to the struggle. In two articles,[7] he subjected Evtushevsky's method to a detailed analysis and harsh criticism, demonstrating the groundlessness of Grube's assumption that all numbers under 100 are accessible to direct "observation" and that all other numbers may be reduced to the first 100 (Lankov, 1951).

3.3 *On Some Pre-Revolution Handbooks for the Elementary School*

In a handbook that went into 25 editions, A. I. Goldenberg (1886) demonstrates that teaching children to perform and apply arithmetical

[6] Leo Tolstoy had spoken out in harsh criticism of Grube's method in 1874. Tolstoy published his *ABC*, which included a section on arithmetic, and he himself taught peasant children. *Editor's note*: On the debates of that time, particularly concerning domestic and foreign methodology, see Karp, A. (2006), "Universal responsiveness" or "splendid isolation"? Episodes from the history of mathematics education in Russia. *Paedagogica Historica*, 42(4–5), 615–628.

[7] Notably, "German ideology in the Russian school" in the journal *Russian News*, 1880, No. 196.

operations is a necessary and sufficient objective of the study of arithmetic at the elementary level:

> While learning calculation techniques the children understand the goal, perceive the means by which they may independently achieve the goal, and are taught to see the decimal number system as a subtle and refined instrument, the full value of which is lost upon us, because it is so simple and so familiar. (p. xii)

Goldenberg asserted that in learning the number system, children acquire mental skills, the value of which far exceeds the limits of performing calculations.

Goldenberg's exercise book for the first grade (Goldenberg, 1903) contains problems and exercises "for the first hundred" (numbers up to 10, "round" numbers, and other numbers up to 100). Their subject matter is "derived from concepts accessible to children and pertains to urban as well as rural environments." The author notes in his introduction that exercises are valuable not only with respect to their arithmetical content but also insofar as they foster in children precise and expressive language. Here are a few sample problems from this exercise book:

- What is the cost of an *arshin* [~28 in] of cloth if a *vershok* [1/16 of an *arshin*] costs 18 copecks?
- $(7 \times 13) - (56 \div 14) + (75 - 67) - 18$
- There are equal numbers of boys and girls in one family. All of the children went out into the forest and collected 57 mushrooms, each boy returning with 7 mushrooms, and each girl with 12. How many children in total are there in the family?
- 24 masons paved a street in 5 days. How many masons would it take to pave the street in 15 days?
- A traveler left a station 5 hours after the luggage cart and followed it along the same route; the cart covered 4 *verst* in an hour, and the traveler covered 10 *verst* in an hour. At what distance from the station would the traveler overtake the cart?

The exercise book includes simple problems (with a single operation) and complex problems (such as deriving the fourth term of a proportion, proportional division, or word problems on motion).

The methodology for teaching arithmetical operations was further advanced by K. P. Arzhenikov[8] (1862–1933), author of *Collection of Arithmetical Problems and Exercises for Elementary Schools* (1898–1917); V. K. Bellustin[9] (1865–1925), author of *Arithmetical Problems* (10 editions before 1919); F. I. Egorov (1845–1915), author of *Collection of Arithmetical Problems* (Egorov, 1895); and S. I. Shokhor-Trotsky (1853–1923), author of *Collection of Exercises in Arithmetic for Public Schools* (1888–1915) and *Methodology for the Teaching of Arithmetic* (1886; about 10 editions).

Here are a few sample problems from Egorov's text (1895), which includes exercises for reproducing calculation strategies along with more creative problems such as replacing omitted digits in operations with large numbers (e.g. #766) and calculating the values of expressions with multiple operations, including problems asking for the most efficient solution (e.g. #927, #1017):

$$\begin{array}{r} 8\,3\,5\,7 \\ \times \qquad 6 \\ \hline 5\ *1*2 \end{array}$$

- $84 \cdot 2 \cdot 30 - 84 \cdot 60$
- $21250 \div 425 + 2975 \div 425 - (21250 + 2975) \div 425$

Egorov paid special attention to the theory behind arithmetical operations. In the chapter titled "Changing the Results by Changing the Terms,"[10] he examined cases where the results of operations with two or more terms changed when some of the terms increased and others decreased; and cases where the results remained the same even when the terms were changed. He made all properties of arithmetical operations consequent upon the students' understanding of such transformations. For example, from the rule concerning the increase of sums, he derived the following conclusion: "In order to add any number to a sum, it is sufficient to add this number to

[8]K. P. Arzhenikov, *Lessons in Elementary Arithmetic* (1898), and *Methodology of Elementary Arithmetic* (through 1935).

[9]V. K. Bellustin, *Methodology of Arithmetic* (1899–1919).

[10]F. I. Egorov, *Methodology of Arithmetic* (fourth edition in 1904, last in 1917).

any one of the addends." Here are a few sample problems (Egorov, 1895):

- A factory employs 240 men and 160 women; another factory has 50 more men and 30 less women than the first. Which factory has more workers in total?
- 5700 + 3800 + 1400.
  [Solution: 6000 + 4000 + 1000 − (300 + 200 − 400) = 11000 − 100 = 10900.]
- An office worker had saved 225 rubles of his salary in the first year; in the second year he was able to save only 199 rubles, even though his salary was increased by 174 rubles. By what amount had his expenses increased in the second year?

The official elementary school curriculum of this period was marked by a certain scarcity of mathematical content and the absence of theory (Pchelko, 1977):

> Counting up to and down from 100. Four operations with numbers from 1 to 20. Introduction to digits and arithmetical notation. Demonstration of the basic arithmetical concepts with illustrations. Roman numeration up to XX.

Nevertheless, many substantive textbooks for elementary schools were published at this time.

Overall, by the end of the 19th century and the beginning of the 20th century, the Russian school had amassed rich experience in teaching elementary-level mathematics. The principle of "visualization" was universally accepted. Each school had an "arithmetical box" and a classroom abacus, and used tables for performing calculations with one- and two-digit numbers. For the majority of students, elementary education was completed in "two or three winters." At this time, education implied two objectives: the material, practical one, and the formative, developmental one. The study of arithmetic was widely thought to have a purely practical aim: students were taught such things as could be useful in everyday activities. Nevertheless, many teachers who "could not abide the 'drilling' associated with practical training" (Shokhor-Trotsky, 1886) demanded that the students draw from their education the full range of intellectual and spiritual experiences.

Shokhor-Trotsky, the founder of the "practical exercise" method, identified a threefold objective to mathematical teaching: educational, formative, and practical. He believed that the educational component is attained when the student has acquired a set of examined mathematical notions, concepts, ideas, and skills. The aims of education are attainable only when students learn willingly and gladly. They must derive both physical pleasure (from producing successful diagrams, calculations, and models) and intellectual pleasure (from completing work, and overcoming difficulties). These ideas are in accord with contemporary views on the psychology of education.

The formative component, according to Shokhor-Trotsky, is attained through the cultivation of "intellectual–cultural" habits. Students must grasp the notion of functional relationships within the limits of their knowledge; must develop powers of observation and a critical attitude toward the veracity of observation; must acquire a habit for precise verbal formulation of questions, generalizations, logical arguments, and so on. The teacher must cultivate the students' interest not only in mathematical knowledge but also in its application in reality (both in school and in everyday life).

Shokhor-Trotsky defined the practical objective as a degree of mastery of mathematical concepts and skills such as befits any cultured person. In his opinion, this so-called "baggage" was of no less importance than the mental skills fostered by elementary mathematical education (Shokhor-Trotsky, 1886).

F. A. Ern (1912) had devoted one of three chapters of his *Notes on the Methodology for Teaching Arithmetic* (pp. 55–58) to the objectives of studying arithmetic: the material and the formal. Material objectives, in his opinion, are attained when students receive information that is valuable in and of itself. Thus, the study of arithmetic "comes down to the study of arithmetical operations, their substance and execution." In order to attain the material objective, one must "teach the children to arrive at the result correctly, promptly and, if possible, elegantly." First, the pupils must learn oral operations with numbers up to 100, then move on to written operations, after which the two must proceed in parallel, neither one supplanting the other. Here, one needs not only problems but also special "number exercises." In order to perform arithmetical operations elegantly, the student must be able "to choose the simplest of available operations, in this case, the one that leads quickest to the result" (Ern, 1912, p. 77). Ern paid special attention to so-called simplified techniques of calculation. He believed that once students become familiar with the basic properties of arithmetical operations and the theorems governing the changes of the results of operations, they must turn to exercises that apply this knowledge to actual calculations, such as:

- $125 \times 36 = 125 \times (4 \times 9) = (125 \times 4) \times 9 = 500 \times 9 = 4500$;
- $96 \div 24 = 96 \div (3 \times 8) = (96 \div 3) \div 8 = 32 \div 8 = 4$;
- $245 + 197 = 242 + 200 = 442$; $245 - 197 = 248 - 200 = 48$.

According to Ern, the study of mathematics aims at a balanced and unified cultivation of the students' skills, intelligence, emotional depth, and willpower. The most important of these is intellectual development: formulation of clear and precise notions and concepts, and acquisition of logical thinking skills. Ern believed that students must arrive at an understanding of number and arithmetical operations, and their properties by way of generalization. The habit of thinking logically and testing the veracity of an assertion by reasoning about it is important in and of itself. It is moreover important to cultivate in students the habit of working independently through solving and especially composition of arithmetical problems. He saw this type of work as the fundamental form of creative activity that rouses interest and entices students toward independence.

The foregoing views of progressive Russian educators on elementary mathematical education — as interesting as they were modern — did not, however, gain wide acceptance. In the 1901 *Courier of Experimental Physics and Elementary Mathematics,* V. V. Lermantov made the claim that the school has a duty to instruct its students in various types of knowledge that are in demand and have direct application in "the everyday struggle." The journal's editor, V. F. Kagan, countered that Lermantov's views hold true for specialized schools only, and that "any nation that permits specialized education to supplant general education is in great peril." Among the diverse skills to be learned, Kagan singled out the most important and the most difficult of all — the skill of thinking. That is the sole objective of general education, to be attained by cultivating in students a coherent worldview and humane attitudes.

The purpose of mathematical education was debated at the two National Congresses of Teachers of Mathematics in Saint Petersburg (winter 1911–1912) and in Moscow (winter 1913–1914). In his presentation, A. G. Pichugin asserted that despite its formal and logical significance, the study of mathematics must be practically useful. "This usefulness is to be understood not in the sense of rank utilitarianism that shuns any thought that cannot be exchanged for ready money, but that pure utility that speaks of the broad horizons of a comprehensive education" (Pichugin, 1913, p. 160). Professor A. K. Vlasov noted that "the objective of mathematical education … is to foster in the pupil a capacity for mathematical reasoning … that addresses itself as much to number and calculation, as to special conceptualization and organization … ." (Vlasov, 1915, p. 25). Participants stressed the importance of pictorial geometry, functional propedeutics, and reasoned calculations for the elementary school curriculum. The initiatives of these conventions were cut short by the First World War.

4 Elementary Education in the Complex Programs of Soviet Russia, 1918–1932

After the October Revolution (1917), in the spring of 1918, the new state issued a Decree on General Education, establishing a unified labor school for all segments of society. The new school comprised two stages: the first lasted five years (later four) for children aged 8–13; the second lasted four years (later five) for children aged 14–17. The curriculum was structured around principles of real-world application, ethnic and gender equity, and instruction in the native language. During this period, mathematics was not taught as a separate subject. Instead, all subjects were oriented toward the study of such complex ideas as "nature and man," "labor," or "society," aimed at cultivating in the pupil a comprehensive view of social reality. The study of mathematics had a strictly practical purpose.

> Acquisition of speech, writing, reading, counting and measurement must be fused with the study of concrete realities; there should be no distinct subjects such as arithmetic or Russian … . (Proekt, 1918).

The "Model Curriculum Project for the Primary Stage of the Unified Labor School–Commune" (Proekt, 1918) states that education shall not be oriented toward communicating maximum knowledge (knowledge without application is useless). The most important goal is for students to work independently on problems encountered in the everyday school environment. "There are nothing but problems requiring mathematical application. Mathematics must spread out its roots and find nourishment wherever there is strict correlation between phenomena, subject to quantifiable analysis" (p. 43). At the same time, the Project of 1918 introduced many new topics, including functional propedeutics, construction of diagrams, finding the area and volume of various figures, and so on. The ideas it set out, however, were not fully realized.

Taking labor as the "axis of existence," the programs consider each phenomenon not discretely, but in relation to other everyday phenomena grounded in the production economy. From year to year, the field of study expands as students grow and develop new skills. The table below gives an overview of the program divided into grades (Lankov and Moshkov, 1927, p. 6).

	Nature and man	Labor	Society
1st grade	Seasons of the year.	Working life of the family (urban or rural).	Family and school.
2nd grade	Air, water, soil. Cultivated plants and animals. Care for these.	Working life of the village or city district, where the child lives.	Social institutions of the village or the city.
3rd grade	Elementary observations in physics and chemistry. The nature of the region. The life of the human body.	Regional economy.	Social institutions of the region. Scenes from the country's past.
4th grade	Geography of the USSR and other countries. The life of the human body.	Economy of the USSR and other countries.	Political system of the USSR and other countries. Scenes from mankind's past.

Regional natural history was the common thread running through the four years of the curriculum. The primacy of direct observation gradually gave way to the predominance of the written word in the form of books, reference works, newspapers, and other materials (Proekt, 1918, p. 9).

Here are some of the topics covered in the textbook for the second grade of a rural school (Zenchenko and Emenov, 1926, pp. 90–91):

> Summer pastimes and summer work for children. Life of the school. Nature in autumn. Famine in the Russian Federation. The human body. Cultural life of the village. Feeding of livestock.

The following are subject headings from the exercise book for the third grade (Lankov, 1926):

> Our village. The October Revolution. Our region. Our town. Man. Our place: district, region. Summer work.

Here is an assignment from the exercise book for the first grade of an urban school (Kavun and Popova, 1930, p. 58):

(a) Measure every day the depth of the snow in the sun and in the shade. Record your readings.
(b) Build two snowmen, each 50 cm tall — one in the sun, the other in the shade. Measure their height every day. Record your readings.

Below are a few problems from the exercise book for the second grade of a rural school (Zenchenko and Emenov, 1926):

- A girl wanted to know how many raspberries she had gathered over the summer. It turned out that in July she had gathered 20 pitchers of red raspberries and 10 pitchers of yellow raspberries, and in August she had gathered 20 pitchers of red raspberries and 30 pitchers of white raspberries. How many jars of raspberries had the girl gathered in her garden? Make up a problem about your own experience of gathering raspberries.
- Draw the path from the village to the forest where you gathered berries and mushrooms.
- As an experiment, some children had taken 100 g of oats and picked out all impurities: there were 14 g in total. How many grams of seeds would remain after impurities had been taken out

of 500 g, 1000 g, and 700 g? Carry out the same experiment with the seeds in your care and make up a problem based on your observations.

- The blood in a healthy human body circulates 120 times in one hour. How many kilograms of blood does the heart pump in this time if the total weight of blood in the body is 5 kg?
- Calculate. Make up similar problems and solve them.

18 + 2	28 + 2	38 + 2	25 + 5
45 + 5	65 + 5	16 + 4	36 + 4
56 + 4	47 + 3	67 + 3	87 + 3

- There are 36 children in the first grade, 30 in the second grade, and 30 again in the third and fourth grades. How many children in total study at the school? Draw a diagram representing the distribution of children in the different grades of your school.
- Draw a plan of the classroom, the school, and the school grounds.
- Build a cubic centimeter and a cubic decimeter using paper and glue.

One positive element of these "complex" curricula was that the study of mathematics was motivated by the demands of the student's everyday life and took into account personal experience. However, the lack of systematic study, simplification of materials, and lack of concern for mathematical skills all contributed to an education that "failed to instill deep and systematic knowledge, and left students largely unprepared for publicly useful activity or further training" (Pchelko, 1977, p. 15).

5 The Study of Arithmetic in the Soviet Elementary School, 1932–1969

The program of study developed after the fall of the "complex" curriculum outlined a precise list of mathematical skills: instances of performing arithmetical operations, types of word problems, elements of the metric system of measurement, fractions, and visual geometry. The instructional material was arranged systematically, broken down by concept and age level. This ensured the material's accessibility within the framework of a uniform mandatory elementary education. Uniform requirements were imposed on all students across the country. This program, with minor adjustments, operated for nearly 40 years[11] (Pchelko, 1977). At this time, the objective of elementary education in arithmetic was defined as the acquisition of knowledge, skills, and experiences necessary for pursuing further school education.

The author of the first "stable" textbooks in arithmetic in the USSR was N. S. Popova (1885–1972). Here are the topics covered in that text under the heading "The first 100":

> Addition and subtraction. Measurement of straight lines and scale drawing. Addition and subtraction of nominative numbers. Comparison by subtracting. Multiplying and dividing by 5, 3, 4, 6, 8, 9, 7. Basic diagrams. Problems with time. Half, quarter, eighth. So many times greater. Multiplication and division by single-digit or two-digit numbers without tables. Comparison by dividing. Division with remainder. Problems and exercises using all operations with numbers under 100. (Popova, 1933)

At this time, calculations were carried out without any sort of theoretical background: students performed simple operations by mimicking an example, while expressions containing multiple operations followed the order of operations (taking parentheses into account). The theory behind the calculations remained "unspoken, not set out in precise language" (Kavun and Popova, 1934, p. 10). Only in 1960 were certain elements of theory introduced at the fourth-grade level.

The predominant teaching method included the teacher's explaining new material, solving problems that targeted newly acquired skills, independent student work, and experimental–practical exercises. Students falling behind in the class were given special attention. Teachers began introducing differentiated assignments, visual aids, and didactic games. For example, half of the teaching materials for the

[11] These adjustments included: increasing the elements of polytechnic training and its practical orientation; reducing somewhat the scope of the program to accommodate younger students entering the first grade at age seven.

fourth grade (Popova, 1961) were made up of board games: "Lotto with Pictures", "Where Is My Place?", "Molchanka" ["Stay Silent!"], etc. The game "Circular Exercises" (one student solves an exercise, but instead of giving the answer outright he or she selects another exercise that begins with the answer to the first one) was given in eight different versions (one of which is as follows: $9-4, 5+5, 10-7, 3+5, 8-6, 2+7$).

Eventually, the view of the elementary school as a place for developing skills only lost its currency. In the 1960s, the Academy of Pedagogical Sciences of the USSR had carried out a major study which revealed significant cognitive abilities in children. A new program of study for grades 1–3 was developed, based on the findings of the Institute for General and Polytechnic Education, the Institute of Psychology, and the Herzen Leningrad Pedagogical Institute under the direction of N. A. Menchinskaya, M. I. Moro (Menchinskaya and Moro, 1965), M. A. Bantova, A. S. Pchelko, and A. M. Pyshkalo. After a lengthy trial period across hundreds of schools, it was approved and implemented.

6 The Elementary Course in Mathematics in the Soviet School, 1969–1990s

In 1969, Soviet schools adopted the elementary course in *mathematics*. According to the well-known methodologist and author of the new curriculum, Pchelko (1977):

> The appearance of such a course, comprising arithmetic, algebraic propedeutics and elements of geometry, is a remarkable achievement, highly rational and absolutely novel, not only in the history of our school system, but also in worldwide practice. The three mathematical disciplines — arithmetic, algebra, and geometry — that have been taught separately for centuries are hereby joined in a synthetic course — the elementary course in mathematics. (p. 17)

The 1969 program grounded fundamental practical skills in theoretical knowledge and was characterized by "the tendency to maximize the students' cognitive abilities and in every way promote their development throughout the educational process" (Programma, 1971, p. 18). Theoretical knowledge became part of the curriculum primarily as a means of explaining calculation techniques. The nature of arithmetical operations, the interdependence of terms and results, multiplication by 0 and 1, and so on, were taught at a fairly high level of abstraction. In many cases, letters were used to generalize statements about numbers. This information was first absorbed with the help of specialized exercises and was subsequently used to explain calculation techniques. M. A. Bantova had developed a system of structuring calculation skills that is still widely used today (by a variety of authors). Students were taught a variety of calculation techniques and given the opportunity to choose the most rational of the lot, such as:

$$48 \cdot 25 = (40 + 8) \cdot 25 = 40 \cdot 25 + 8 \cdot 25$$

$$48 \cdot 25 = (12 \cdot 4) \cdot 25 = 12 \cdot (4 \cdot 25)$$

$$48 \cdot 25 = 48 \cdot (20 + 5) = 48 \cdot 20 + 48 \cdot 5$$

$$48 \cdot 25 = 48 \cdot 100 \div 4 = 48 \div 4 \cdot 100$$

For each new calculation technique, students were given a theoretical explanation and asked to perform exercises so as to secure the new skill. Here are a few examples of such exercises drawn from a contemporary textbook modeled on the exercises of that era (Moro *et al.*, 2009; third grade; pp. 6–9):

- Calculate, then explain your calculations:

$$(5 + 3) \cdot 4 \quad (20 + 7) \cdot 2 \quad (6 + 4) \cdot 8$$

- Solve the problem using different methods:
  A grandmother gave each of her three grandchildren 4 red apples and 4 yellow apples. How many apples in total did the grandchildren receive?
- Explain why these equalities are correct:

$$8 \cdot 3 + 7 \cdot 3 = (8 + 7) \cdot 3 \qquad 6 \cdot 8 + 4 \cdot 8 = 10 \cdot 8$$

$$17 \cdot 5 + 3 \cdot 5 = (17 + 3) \cdot 5$$

The topic "Changing the results by changing terms" was gradually covered in grades 1–3. At the first stage (grades 1 and 2), students learned about the type of change that occurs when one of the terms is altered; for example, if the value of one addend is increased while the other remains the same, the sum will also increase. At the second stage (third grade), students were able to quantify the change and formulate rules; for example, if the value of one of the addends is increased by a certain number of units while the other remains the same, the result will be increased by the same number of units.

Here are a few sample exercises for the first and second stages from textbooks for the first and third grades:

- Fill in the blanks with any appropriate number (first grade):

$$15 + 3 > 15 + \cdots \qquad 17 - 5 < 17 - \cdots$$

$$45 + \cdots > 18 + 45 \qquad 68 - \cdots < 68 - 5$$

- Calculate the value of the second expression using the value of the first (third grade):

$$420 \div 6, \quad 420 \div (6 \cdot 2)$$

$$320 \div 8, \quad 320 \div (8 : 2)$$

$$540 \div 6, \quad 540 \div (6 \cdot 5)$$

The students' grasp of these properties served as the foundation for learning specific calculation techniques (e.g. $368 + 99 = 368 + 100 - 1$) and as the first steps toward an understanding of functional dependency.

This approach encouraged conscious, rational, and accurate calculation, and promoted a cognitive development and calculation culture among elementary school students.

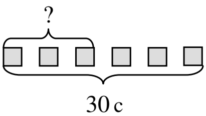
Through algebraic propedeutics, students learned about expressions (with numbers and variables), equalities, inequalities, and equations, solution strategies for word problems, and functional dependencies of quantities.

Here are a few sample problems from textbooks of that period (Moro *et al.*, 1970, pp. 175, 219):

- Compose exercises and solve them: $17 + x = 20$; $x + 3 = 20$; $17 + 3 = x$.

- Find the value of $4 \cdot d$ if $d = 5, d = 6, d = 8, d = 3$.
- Six envelopes cost 30 copecks. How much do three envelopes cost?

Pictures and tables accompanied new problems, as in Fig. 1.



Price	Quantity	Cost
Same	6	30 c.
	3	?

Fig. 1.

(Later, when solving similar problems, students would draw up tables of their own.)

Plane geometrical figures were studied in all three years of the elementary school: segment, broken line, types of angles, and polygons — triangles and quadrilaterals, including rectangles (and squares). Students were asked to identify, construct, and transform figures (see e.g. Moro *et al.*, 1970, p. 213):

> In the diagram below, find 2 pentagons, 2 quadrilaterals, and 2 triangles. In addition, find 6 right angles.
> Cut out these figures and use them to construct new figures.

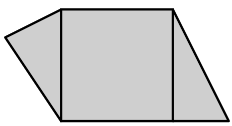


Fig. 2.

Textbooks gave special attention to exercises requiring comparison and analysis, concretization and generalization, independent work, and creativity. After a trial period in Russian schools, the textbooks were translated into the languages of 11 Soviet republics.

Two other programs in elementary mathematical education appeared at the same time, created by L. V. Zankov and V. V. Davydov. They were used only in an experimental setting.

Davydov's program[12] placed primary emphasis on construing the child as a subject of educational activity and developing his or her theoretical thinking through a deductive approach to the structuring of an elementary course in mathematics. Primacy of place was given to the study of magnitudes, which, through comparison and practical measurement, yielded number; the course made use of a generalized notation to describe relations of magnitudes.

The system of Zankov[13] (1901–1977) aimed at maximizing students' general development and was based on the following principles: intensive development of all children; systemic and comprehensive content; primacy of theoretical knowledge; demanding, fast-paced instruction; making the child aware of the educational process; tying the educational process to the child's emotional life; problematization and variability of the educational process; and individualized approach.

These systems worked well in terms of general development, but — according to their critics — were inefficient in furnishing students with specific mathematical skills.

The widely adopted program and textbooks of Moro, Bantova, and Beltiukova were new and unfamiliar to teachers. Despite the tremendous effort through a variety of publications (Bantova *et al.*, 1976) to explain the methodological ideas that informed the program, not all of them would be realized in general practice. Over time, the program went through numerous changes, which eventually undermined somewhat the original emphasis on general development, reduced the role of theoretical knowledge, and underscored practical application by increasing the number of practical exercises. The authors of the textbook later wrote: "The changes made to the program in mathematics over the past few years pursued a very important goal: to give the course a more practical orientation" (Kolyagin and Moro, 1985, p. 3).

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[12]For a detailed account of Davydov's system, see http://www.centr-ro.ru/school.html

[13]For a detailed account of Zankov's system, see http://www.zankov.ru

After a series of experiments in the 1980s, textbooks were rewritten to accommodate a four-year elementary education for children entering school at the age of six (M. I. Moro, M. A. Bantova, and G. V. Beltiukova; edited by Yu. M. Kolyagin).

It should be noted that the textbooks of Moro *et al.*, which had regrettably lost some of their developmental potential through simplification, became the foundation for subsequent educational programs and served to acquaint the average elementary school teacher with their principles. The present author believes that the educational program created by Moro *et al.* (especially prior to its major simplifications) offered a thoroughly reasoned and structured system of mathematical education (which may not hold true for other programs).

7 Elementary Mathematical Education in Russia, 1990s

7.1 *Fundamental Program Requirements and Characteristics of Contemporary Textbooks*

Following the social democratization of the 1990s, alternative educational programs gained official recognition alongside that of Moro *et al.*; these included both Zankov and Davydov, as well as N. Ya. Vilenkin, L. G. Peterson, and N. B. Istomina (Programmy, 1998). The appearance of competing programs in general education and preschool training led to the development of an Educational Standard and the Conception of the Content of Continuous Education.

The Educational Standard for Russian Schools (Uchebnye standarty, 1998) acknowledges the changing role of mathematics in general culture and education. The Conception of the Content of Continuous Education sets out the following objectives for elementary mathematical education:

- Development of the basic forms of intuitive and logical thinking and mathematical language; development of intellectual operations (analysis, synthesis, comparison, classification, etc.); ability to operate with symbolic systems;

- Command of a specific system of mathematical notions and common operations;
- A basic grasp of the leading mathematical method for understanding the physical reality — mathematical modeling (Kontseptsiya, 2000, pp. 16–17).

The Federal Educational Standards (second generation), set to take effect in 2010, lay out performance requirements, structural guidelines, and conditions (staffing, financial, technical, material, etc.). Standards consider subject-specific performance alongside metadisciplinary and personal accomplishments. *Personal* accomplishments include readiness and capacity for self-development, motivation for study and acquisition of knowledge, system of values, and foundations of civic identity. *Metadisciplinary* accomplishments include universal learning operations (which form the basis of learning capability) and interdisciplinary notions. *Subject-specific* performance includes acquisition and application of new subject-specific skills, as well as a system of elements of scientific knowledge at the basis of the contemporary scientific understanding of the world.

Subject-specific performance requirements in Mathematics and Informatics include the following (Ministry, 2009, pp. 12–13):

- Using mathematical skills to describe and explain objects, processes, and phenomena, and to evaluate their quantitative and spatial characteristics;
- Receiving the foundations of logical and analytical thinking, spatial imagination and mathematical speech, measurement, enumeration, estimation and assessment, visual representation of data and processes, notation and performance of algorithms;
- Basic experience in using mathematical skills to solve theoretical and practical problems;
- Ability to perform arithmetical operations (oral and written) with numbers and numeral expressions; solve word problems; follow an algorithm and construct basic algorithms; examine, identify, and reproduce geometrical figures; work with tables, diagrams, graphs, sequences, and populations; visualize, analyze, and interpret data;
- Basic notions of computer literacy.

The main objectives of the study of Mathematics and Informatics, according to the same Standards, are as follows: development of mathematical speech, logical and algorithmic thinking, imagination, and preliminary notions of computer literacy (p. 22).

At this time, there are 15 curriculum "series" or "complexes,"[14] as they are called, in Russia in mathematics for the elementary school, evaluated and included in the federal register of textbooks recommended by the RF Ministry of Education and Sciences for use in Russian schools (see http://www.edu.ru/db-mon/mo/Data/d_09/m822.html).

Different methodological ideas underlie the various "complexes;" however, all of them give primacy of place to the developmental aims of education. Ivashova *et al.* (2009) stress the equal importance of developmental and discipline-specific aims.

All of the "complexes" break down the material according to the basic components of learning activity (positing an objective, proposing ways of attaining the objective, planning, following the plan, self-monitoring and self-evaluation, reflection). Bashmakov and Nefedova (2009) and Ivashova *et al.* (2009) include an overview at the start of the textbook (section titled "What will we learn?"), quarterly review sections, and reference materials.

Several textbooks make use of creating so-called "problem situations" in the material: for example, in Ivashova *et al.* (2009) and Istomina *et al.* (2009), students are asked to evaluate solving strategies, explain underlying reasoning, choose the best option, and find and correct errors. In Ivashova *et al.* (2009), correction of errors presupposes variability, as in the following exercise:

Check the calculations. Correct one of the terms or the final value.

–	4	8			–	8	2			–	6	6			–	9	0
	3	1				1	7				2	9				5	6
	7	9				6	4				4	7				4	4

[14]Typically, such a "complex" includes not only textbooks, but teachers' manuals, problem books, and other supplemental materials.

Arginskaya *et al.* (2009), Davydov *et al.* (2009), and Istomina *et al.* (2009), among others, encourage students to find a solution strategy independently and draw their own conclusions. Exercises involving "problem situations," composition, or transformation of existing problems, numbers, expressions, investigative exercises, and so on promote creative thinking in students. For example:

What is the rule governing the transformations of the original expression in each column?	$7 \cdot 4 + 18 - 9 \cdot 3$	$86 - 7 \cdot 3 - 49 \div 7$
	$28 + 18 - 9 \cdot 3$	$86 - 21 - 49 \div 7$
	$28 + 18 - 27$	$86 - 21 - 7$
	$46 - 27$	$65 - 7$

Use the same rule to construct a new column beginning with the expression $9 \cdot 5 - 6 \cdot 4 \div 8$ (Istomina *et al.*, 2009, 3rd grade).

The following strategies reflect the movement toward personalized education:

- Students are asked to characterize exercises as easy or difficult, interesting or boring, to choose the most comfortable solution strategy (Alexandrova, 2009; Davydov *et al.*, 2009; Ivashova *et al.*, 2009), explain the solution process (Alexandrova, 2009; Arginskaya *et al.*, 2009; Ivashova *et al.*, 2009), compose an original exercise and teach it to others (Alexandrova, 2009), and compose problems based on personal observations (Davydov *et al.*, 2009).
- Exercises are worded in a personalized manner: "Do you know … ?" "How much would you have to spend if you wanted to buy … ?" "Draw up a plan of action and tell it to others" (Ivashova *et al.*, 2009; Rudnitskaya and Yudacheva, 2009).
- Emphasis is placed on alternative solving strategies (Alexandrova, 2009; Istomina, 2009; Ivashova *et al.*, 2009), and on choosing the most appealing exercises (Rudnitskaya and Yudacheva, 2009) and solving strategies (Ivashova *et al.*, 2009).
- Exercises of varying difficulty include advanced-level (Ivashova *et al.*, 2009), required, and supplemental exercises (Bashmakov and Nefedova, 2009; Moro *et al.*, 2009; Rudnitskaya and Yudacheva, 2009).

One of the peculiarities of Ivashova *et al.* (2009) is psychological differentiation. Exercises intended for students with different styles of perception and information processing are marked accordingly: sign  stands for kinesthetic perception (exercises dealing with movements and notions about movement), sign  stands for visual perception (exercises involving images and diagrams), aural perception (listening), and sign  stands for verbal representation. Here are some exercises for the derivation of the number 5 (first grade):

 Lay out four circles. Add one more. How many circles are there?

 Examine the drawing and explain how the number 5 was derived.

 Name four girls, then name another one. Now say the five names all together.

Typically, textbooks break down the material into discrete lessons; the exceptions are Alexandrova (2009), Istomina (2009), and Rudnitskaya and Yudacheva (2009), where the material is presented thematically. Rudnitskaya and Yudacheva (2009) include a review section after each theme, while Davydov *et al.* (2009) and Alexandrova (2009) gather all the review materials into a single section at the end of the textbook, titled "Check your skills and knowledge" or "Check yourself!". In providing review sections, textbooks encourage self-monitoring by students.

Several sections are aimed at broadening or deepening the students' mathematical skills, e.g. "This is interesting!" (Alexandrova, 2009; Rudnitskaya and Yudacheva, 2009), "Problems for those who like to work hard" (Alexandrova, 2009), "For the math enthusiast" (Demidova *et al.*, 2009), "From the history of mathematics" (Bashmakov and Nefedova, 2009; Ivashova *et al.*, 2009; Rudnitskaya and Yudacheva, 2009), "Let's play with the kangaroo" (Bashmakov and Nefedova, 2009), and so on. Here is a sample exercise from the third-grade textbook of Bashmakov and Nefedova (2009):

> Which number matches the following description? It is even, none of its digits are the same, and the digit in the third position is double that in the first position. (A) 1236, (B) 3478, (C) 4683, (D) 4874, (E) 8462.

We should note that for the enrichment of their courses, teachers of elementary mathematics frequently make use of the publication by Kalinina *et al.* (2005), which essentially doubles as an encyclopedia for the elementary school, written in accessible language.

Many textbooks include group exercises, aimed at developing communication skills in children. Working in dialog, the students acquire new skills and knowledge and learn to accept another's point of view. For example, Istomina (2009), Ivashova *et al.* (2009), and Rudnitskaya and Yudacheva (2009) make use of recurring characters with competing viewpoints, which are sometimes correct and sometimes incorrect.

A number of texts include reference materials (such as average speeds of various types of transportation and animals, or weights of various types of objects and materials), which train the students' ability to work with data and promote interest in mathematics and creativity in composing one's own exercises.

Many of the "complexes" place special emphasis on the cultural aspect of mathematics through word problems (including problems with interdisciplinary content) and calculation exercises that require students to decipher certain names, terms, etc., contextualize numerical data, and identify geometrical figures in their immediate surroundings or in architectural structures. In certain textbooks, entire lessons are structured around a narrative. For example, the review sections in Bashmakov and Nefedova (2009) for the second and third grades have unifying themes: "Little Boy and Karlsson" (recalling Astrid Lindgren's story), "A Flight to the Moon," and "The Golden Fleece."

For example, a calculation exercise in "A Flight to the Moon" asks the student to decipher the name of the first astronaut to step on the surface of the moon, which requires a series of calculations to determine the correspondence between numbers and letters.

Overall, many of the "complexes" in elementary mathematics may be characterized as "next generation." Their content is primarily scientific, personalized, and aimed at general development, follows the "active" approach to elementary education, and generally conforms to current standards (Uchebnye standarty, 1998) and forthcoming educational standards (http://standart.edu.ru/catalog.aspx?CatalogId=531).

7.2 *The Content of the Elementary Course in Mathematics*

Let us consider the content of the elementary course in mathematics across a variety of textbooks.

All of the textbooks cover the following major topics: "Numbers and arithmetical operations," "Solving arithmetical problems," "Magnitudes," "Elements of algebra," and "Elements of geometry." Some of the textbooks include additional topics such as "Elements of combinatorics and elements of logic" (Demidova *et al.*, 2009; Ivashova *et al.*, 2009; Peterson, 2009; Rudnitskaya and Yudacheva, 2009); "Elements of descriptive statistics and basic concepts in probability theory" (Demidova *et al.*, 2009); "Unconventional and recreational problems" (Bashmakov and Nefedova, 2009; Demidova *et al.*, 2009; Ivashova *et al.*, 2009; Moro *et al.*, 2009); and "Geometric transformations" (Chekin, 2009; Istomina, 2009; Peterson, 2009). Let us examine some of these topics in greater detail.

7.2.1 *Numbers and arithmetical operations*

All of the textbooks cover the following subjects:

> Counting objects. Names, succession, and notation of numbers from 0 to 1,000,000. Number relations, such as "equal," "greater than," "less than," and their notation: $=, <, >$. The decimal numbering system. Classes and digit positions. The positional principle of number notation.

All textbooks, with the exceptions of Alexandrova (2009) and Davydov *et al.* (2009), are structured concentrically: the students first learn the numbers 1 through 10, then the numbers up to 100, and then up to 1000 and beyond. This corresponds to the child's experience and to the methodological tradition in Russia. The textbooks present a variety of methods for deriving numbers: counting, addition, and subtraction of 1, measurement, and arithmetical operations with other numbers. In Alexandrova (2009) and Davydov *et al.* (2009), the main method of deriving numbers (natural as well as rational, etc.) is measurement. By introducing a variety of measuring units, the course prepares first and second graders for the study of different counting systems and permits them to see the decimal system as one possibility among several.

This approach does not take into account the child's preschool experience, but it permits students with different levels of preparation to feel confident in discovering new knowledge. One drawback of both Alexandrova (2009) and Davydov *et al.* (2009) is that the text does not differentiate between notations referring to magnitudes and those referring to sets or figures. This approach may lead to confusion over such concepts as "finite set" and "size of finite set," "segment," and "length of segment; it runs counter to the principle of continuity, since at a later stage the student will be asked to differentiate these concepts through notation (Beltiukova *et al.*, 2009).

All other textbooks use a concentric structure to teach derivation of numbers, their names and sequencing, the decimal order, positional notation, and various methods of number comparison. Many of the textbooks make use of historical references (Bashmakov and Nefedova, 2009; Demidova *et al.*, 2009; Ivashova, 2009; Peterson, 2009; Rudnitskaya and Yudacheva, 2009).

All textbooks make extensive use of various types of modeling. For example, in learning the decimal numbering system, students are asked to use sticks and bundles of sticks or squares for ones, strips for tens, and large squares for hundreds. The great majority of the textbooks make use of the number line; Arginskaya *et al.* (2009) and Istomina (2009) use the segment of natural numbers, while Alexandrova (2009) and Davydov *et al.* (2009) discuss various kinds of positional notation.

All of the textbooks cover the following subjects associated with arithmetical operations:

> Addition and subtraction, multiplication and division, corresponding terminology. Tables of addition and multiplication. Number relations, such as "greater by … ," "smaller by … ," " … times greater," and " … times less." Division with remainder. Arithmetical operations with zero. Determining the order of operations in numerical expressions. Finding the value of expressions with parentheses and without. Changing the order of addends and multipliers. Grouping addends and multipliers. Multiplying a sum by a number, and a number by

a sum. Dividing a sum by a number. Oral and written calculations with natural numbers. Using the properties of arithmetical operations in calculations. Finding an unknown component of arithmetical operations. Strategies for checking calculations. Solving word problems (with one or multiple operations) by arithmetical means (using a variety of models: schematic drawings and graphs, tables, and shorthand notations with keywords). Relations of proportional magnitudes (velocity, time, distance traveled; price, quantity, cost, etc.).

As for other subjects, let us note that we do not see the advantages of a detailed study of decimal fractions — i.e. construction, rounding, comparison, performing arithmetical operations, and deriving fraction from number and number from fraction — at the fourth-grade level (Alexandrova, 2009). In moving this material from the fifth- and sixth-grade curriculum into elementary school, the textbook runs counter to the principle of succession and shifts attention from other important topics (for example, at the second- and third-grade levels, this textbook has far too few exercises with geometrical figures).

We are also skeptical about the accessibility for students of proportions characterizing work, movement, and buying–selling (Davydov *et al.*, 2009), in order to understand which students need to master such concepts as "process," "event," "variable characteristics," "additional conditions," "uniform process," "variable process," and "speed of a uniform process." In the corresponding textbooks for the fourth grade, one reads: "The speed of the uniform process K indicates the rate of increase of Y with respect to X. $X_1 = X_2, Y_1 > Y_2, K_1 > K_2$." And further on: "The speed of a uniform process is a constant. It indicates how many units of Y correspond to a single unit of X" (Davydov *et al.*, 2009, Book 1, p. 109).

It is interesting to note the use of the calculator (Chekin, 2009; Istomina, 2009; Moro *et al.*, 2009) not as a substitute for manual calculation, but as a way of verifying results. Here are a few sample exercises:

- Find the value of the expressions $37 + 24 - 24$, $52 + 37 - 37$, and $83 - 18 + 18$.

In what ways are these expressions similar? What conclusions can you draw? Verify your results with the help of a calculator, using different numbers. (Istomina, 2009, 2nd grade)

- Using a calculator, add 1, 2, 3, and 4 to the number 372. Which digit changes in the number 372? What other numbers could you add to 372 without changing any of the other digits in the number? (Istomina, 2009; 2nd grade)
- Using a calculator, find out whether the greatest three-digit number is a multiple of the greatest six-digit number. (Chekin, 2009; 3rd grade)
- The value of what expression would you be calculating if you pressed the following sequence of buttons on your calculator?

$$\boxed{2}\ \boxed{3}\ \boxed{8}\ \boxed{9}\ \boxed{7}\ \boxed{7}\ \boxed{-}\ \boxed{2}\ \boxed{3}\ \boxed{8}\ \boxed{9}\ \boxed{0}\ \boxed{5}\ \boxed{\div}\ \boxed{9}$$

(Chekin, 2009, 3rd grade)

The introduction to algorithms in several of the textbooks includes: analyzing existing algorithms; constructing new algorithms (including "everyday life" algorithms — crossing the street, lighting a fire, etc.); types of algorithm notation — verbal and flowchart; and performing calculations using a flowchart. This addresses the requirements set out in the new standard.

A number of "complexes" pay special attention to estimating value, evaluating results, and verifying results. For example, Ivashova *et al.* (2009) include the following assignment:

Calculate and verify using a different calculation technique, such as:

$100 \div 4 = (80 + 20) \div 4 = \square$	$60 \div 4$	$90 \div 5$
$100 \div 4 = (120 - 40) \div 4 = \square$	$14 \cdot 5$	$38 \cdot 2$

Bashamkov and Nefedova (2009) have the following:

Choose the answer out of three given values without performing the calculation to the end. Then evaluate the value and compare it with your choice.

Find the value and compare it with your choice.

(a) 173 + 264 + 435, (b) 236 + 312 + 422, (c) 329 + 119 + 449

(872) (972) (899) (772) (970) (997) (874) (890) (897)

Moro *et al.* (2009, 4th grade, p. 85) have the following:

Pick out the wrong answers without doing the calculation. Solve and check your answer through multiplication.

$$7380 \div 9 = 82, \quad 3010 \div 5 = 62, \quad 56014 \div 7 = 8002.$$

Some of the textbooks include subjects not covered in the standard: "Common fractions, addition and subtraction of fractions with the same denominator, multiplication and division of fractions," "Positive and negative integers," and "Percent."

As far as calculation techniques are concerned, let us note the following: the majority of the textbooks first teach oral calculations and then written calculations. Davydov *et al.* (2009) first introduce the digit-position principle of written calculation and only later ask students to compose and memorize a table of addition (and later multiplication) and learn the techniques of oral calculation. It seems advisable to encourage students to calculate orally whenever possible. Rudnitskaya and Yudacheva (2009) give primacy of place to written calculation, which seems to us a doubtful approach, since in everyday life one is often called upon to calculate "in one's head."

7.2.2 *Arithmetical problems*

Students learn to analyze the problem, establish connections between magnitudes, determine the number and type of operations necessary for solving the problem, choose and explain their choice of operations; to solve the problem using arithmetical methods (in one or two, or even three or four steps), including proportional magnitudes; and to find multiple solutions to the same problem.

Some of the curriculum "complexes" — see e.g. Moro *et al.* (2009) — use basic problems to demonstrate the concrete meaning of operations and to teach concepts such as "by certain amounts/certain times greater/less than," properties of operations, and so on. Problems are introduced gradually, and every time a problem is first given with sets, then with magnitudes, and finally with abstract numbers. In other "complexes" — see e.g. Istomina (2009) — students first learn the skills described above, then apply them to specific problems. In Peterson (2009), Davydov *et al.* (2009), and Alexandrova (2009), all basic problems solved by addition or subtraction are illustrated with schematic graphs and explained as a relation of whole and parts. All textbooks (beginning with the first and second grades) include compound problems, and — starting in the third grade — problems with proportional magnitudes. Children are frequently asked to look for alternative solutions to the same problem. For example, Demidova *et al.* (2009, 4th grade, pt. 1, p. 88) include the following problem:

> How many different ways can you find of answering these questions?
>
> To travel 80 km along a river in a motorboat, one needs 160 L of gasoline. How many liters of gasoline does one need to travel 40 km more? How much farther can you travel if you have 20 L of gasoline more?

Many textbooks contain problems with missing or extraneous information, with data given using letters rather than numbers, and with exercises involving problem change and problem composition.

7.2.3 *Magnitudes*

Following general requirements, all textbooks cover these topics:

> Comparing and ordering objects in accordance with various attributes: length, mass, volume. Units of length (1 mm, 1 cm, 1 dm, 1 m, 1 km), mass (1 g, 1 kg, 1 cwt, 1 ton), volume (1 L), time (1 s, 1 min, 1 h, 1 day, 1 week, 1 month, 1 year, 1 c). Measuring the length of a segment and constructing a segment of a given length. Calculating the perimeter of a polygon. Area of a geometrical figure. Units of area (1 cm^2, 1 dm^2, 1 m^2). Calculating the area of a rectangle.

Other magnitudes or groups of magnitudes related by proportionality are studied in the context of solving word problems.

When working with magnitudes, students are given visual representations of each magnitude and each unit of magnitude. Emphasis is placed on the decimal relationship between geometrical magnitudes (length, area) and mass, as well as on performing operations with numbers signifying magnitudes. Special attention is given to finding the perimeter and area of rectangles. Here are a few sample problems:

- Compare: 7200 m and 72 km, 300,000 m^2 and 1 km^2, 2 h and 80 min, 8 cwt and 740 kg.
- Draw a rectangle with sides equal to 1 dm and 1 cm. Find its area and its perimeter.
- Calculate: 12 m 86 cm + 3 m 45 cm; 45 tons 275 kg − 18 tons 130 kg. (Moro *et al.*, 2009, 4th grade, part 1, pp. 48, 54, 67)

Additionally, some of the textbooks consider archaic units and measurements; the area of a right triangle; volume, units of volume; and magnitudes of angles.

7.2.4 *Geometrical content*

The following topics belong in this section:

> Identifying and reproducing geometrical figures: point, line, segment, angle, polygon — triangle, rectangle (square), their properties, diagonals in a rectangle. Plane figures: types of angles, types of triangles (right, acute, obtuse, isosceles, equilateral), broken line, circle (center, radius, diameter).

In accordance with the new standard, the following geometrical solids have been introduced into the curriculum: parallelepiped, pyramid, cylinder, and cone. At this time, only some of the programs study these figures.

The following types of exercises are in use when studying these topics: identifying figures (choosing one figure among several, from a complex diagram, in the students' surroundings), comparing figures, measuring figures, reproducing figures (on square paper and unruled paper), partition and transformation of figures (by cutting, folding, drawing, mentally), building models of figures (using clay or cutouts), and analyzing surfaces (touch it!). Assignments involving geometrical figures often presuppose practical tasks.

Dorofeev and Mirakova (2009), Ivashova *et al.* (2009), and Istomina (2009) give special attention to the development of spatial imagination or varying the reference point. For example, Ivashova *et al.* (2009) have two seemingly identical exercises with what turns out to have different answers (Fig. 3):

1. What is to the left of the square?
2. What is to the left of the girl?

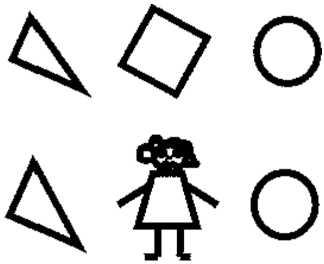


Fig. 3.

Additionally, some of the textbooks study isometry: axial and point symmetry; translation.

The inclusion of such advanced topics as "coordinate plane," "graphs," "dividing circumference into equal parts," and "dividing a segment into 2, 4, 8 equal parts using a compass and ruler" (Rudnitskaya and Yudacheva, 2009) seems to us unjustified. These skills are never used in elementary school.

7.2.5 *Elements of algebra*

All programs study equalities and inequalities (beginning in the first grade), numerical expressions and expressions with variables, equations (typically beginning in the first and second grades; but, in Istomina (2009), only in the fourth grade), and elements of functional propedeutics. The following exercise may serve as an example:

> Substitute appropriate numbers for letters. Solve. Compare values. $a - b \div c$ and $(a - b) \div c$. What numbers can take the place of a, b, c, and what numbers cannot? (Alexandrova, 2009, 4th grade, pt. 1, p. 147)

Additionally, some of the textbooks look at solving problems by means of equations, solving equations based on the properties of equalities, and complex form equations. For example, Arginskaya *et al.* (2009) include the following exercise for the third grade:

> Compare the equations. What is the difference between the equations in the right column and the left column?

$12x - x - 55 = 0$	$2 \cdot (y - 15) + 8y = 5$
$5 + 6a + 4a = 95$	$2 \cdot (x + 3) + 5 = 17$
$3 \cdot (x - 1) + 12 = 18$	$(k + 3) \cdot 5 - 34 = 31$

7.2.6 *Elements of combinatorics*

Several textbooks consider combinatorial problems (finding commutations, permutations, or combinations), solving them either directly by enumeration or using tables and graphs. Here are some examples:

- You have the following products to prepare a breakfast: banana, coconut, baked potato, fish. How many different breakfasts consisting of two dishes will you be able to put together? (Bashmakov and Nefedova, 2009)
- Write down all possible three-digit numbers composed of the digits 3, 5, and 0. (Moro *et al.*, 2009)
- Masha, Vika, Alla, and Tania call each other before a trip. How many phone calls did they make if every girl spoke once to every other girl? (Ivashova *et al.*, 2009)

The last problem may be solved with the help of a table or a diagram:

First way

	Masha	Vika	Alla	Tania
Masha		M-V	M-A	M-T
Vika			V-A	B-T
Alla				A-T
Tania				

Second way

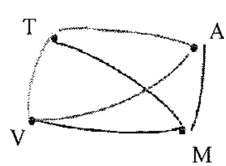


Fig. 4.

7.2.7 *Elements of logic, set theory, modeling*

Some of the curriculum "complexes" consider elements of set theory, including such concepts as sets, member of a set, ways of defining a set, intersection of sets, union of sets, and properties of sets (Dorofeev and Mirakova, 2009; Ivashova *et al.*, 2009; Peterson, 2009). Some of the textbooks take up construction of basic logical expressions using the words "and," "or," "if… , then… " (Alexandrova, 2009; Istomina, 2009; Ivashova *et al.*, 2009; Peterson, 2009; Rudnitskaya and Yudacheva, 2009). Students are asked to compose expressions of various types and determine their truth value.

Many of the textbooks contain exercises aimed at developing mental operations, including exercises involving comparison, analysis, classification, generalization, concretization, and pattern detection (especially in tables of addition and multiplication).

It should be noted that all "complexes" make use of modeling. The ability to use different kinds of models and to express information in different languages (figural, graphic, symbolic, and verbal) not only helps in grasping the principles of elementary-level mathematics but also develops in the student an understanding of the mathematical method of learning about the world — mathematical modeling — which conforms to the requirements of the educational standard (http://standart.edu.ru/catalog.aspx?CatalogId=531).

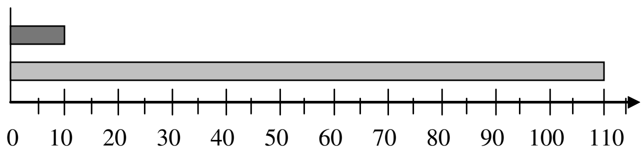
Accordingly, Alexandrova (2009) asks students not only to analyze existing models but also to construct their own, or reconstruct magnitudes from graphic and symbolic models (formulas). All textbooks use diagrams (circles, squares), diagram drawings, and tables to model relationships between magnitudes. Rudnitskaya and Yudacheva (2009) use graphs. Virtually all textbooks make extensive use of modeling when dealing with arithmetical problems.

7.2.8 *Working with data*

The "second-generation" educational standard introduces a new topic: "Working with data." As of now, it does not yet appear in every textbook. According to the standard, a student must be able to read simple worksheets and fill them in with data, and read simple bar charts. A graduating student must have had the opportunity to learn to read simple pie charts and to complete bar charts; recognize identical information presented in a different form; gather and present information in the form of worksheets and diagrams; and interpret information gathered through basic research (explain, compare, and generalize data; draw conclusions and make predictions).

All curriculum "complexes" include some type of assignments involving worksheets; several textbooks (Chekin, 2009; Demidova, 2009; Ivashova *et al.*, 2009; Peterson, 2009) have introduced extensive data analysis, including work with diagrams, such as:

> Use the information in the following diagram to compose and solve a problem with comparison (Chekin, 2009):



Concerning computer literacy for elementary school pupils, as outlined in the educational standard, let us note that grades 4 and 5 study "Informatics" as a separate subject, and that various types of computer-based study supplements — both topic-specific and more general — have been developed and are currently in use. The author of the present chapter is responsible for the mathematical component of the integrated learning "complex" — "Discovering the Laws of Language, Mathematics and Nature" — for grades 1–4. The materials have been evaluated by experts in a variety of disciplines and subsequently made available online through the Consolidated Digital Educational Resources of the RF Ministry of Education (http://school-collection.edu.ru/catalog/pupil/?class=42). The integrated "complex" may be used with any of the existing curricula, but naturally its general approach matches that of Ivashova *et al.* (2009). In addition to testing materials, the online "complex" may be used to find new information and consolidate previously acquired skills. Other textbooks offer supplemental exercise CD-Roms or online components (e.g. http://soft.mail.ru/download_page.php? id=413226 &grp=63164).

8 Conclusion

This chapter offers a brief overview of the history and the present state of mathematical education at the elementary school level. It appears that historical tradition bears out the 19th century methodologist's conception of elementary mathematical education as the study of arithmetical operations with whole positive integers. In the late 20th century to the early 21st century, we have seen an expansion of mathematical content at this level through the inclusion of elements of geometry (including geometrical solids), elements of algebra, and stochastics. It is worth noting that the idea of including solids in the study of geometry figures was put forward as early as 1911–1912 at the 1st Congress of Teachers of Mathematics. Meanwhile, research in psychology has confirmed that the elementary school age is a crucial period in the development of a child's spatial imagination. Today's interest in investigative problems and data processing and visualization — as exemplified in the new educational standard — owes a debt to the early years of Soviet educational practice. There has been a great deal of change in educational methodology, especially when compared to the educational methodology of the 18th century, while progressive ideas of 19th century methodologists, discarded at the time, are being implemented today. Priority is given to methods that champion active development. Mathematical education undoubtedly plays an important role in the overall education and development of elementary school students, and bears directly on their accomplishments in specific disciplines as well as their personal and metadisciplinary achievements.

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3

On the Teaching of Geometry in Russia

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1 Introduction

Perhaps the most striking difference between the teaching of mathematics in Russia and standard mathematics education in the West is that the former includes a separate course in geometry taught over a five-year period. It has been over 50 years since it was declared in the West that "Euclid must go" (cited in Fehr, 1973). Even aside from this, the "Western" course in geometry was often — and continues to be — conceived of as occupying only one year and certainly not as constituting a constant accompaniment for students from sixth grade on, throughout all of their middle and high school years.

In Russia, Euclid and Euclidean geometry did not go anywhere. Plane geometry is taught in grades 7–9 (6–8)[1] for 2–3 hours per week;

[1]We remind readers that after Russian education officially switched to an 11-year program in the early 1990s, the nomenclature changed: sixth grade became seventh grade, seventh grade became eighth grade, and so on.

three-dimensional geometry is taught in grades 10 and 11 (9 and 10), usually for 2 hours per week. The course in plane geometry is thus intended to occupy over 200 hours of classes, and the course in three-dimensional geometry approximately 140 hours. In addition, the mathematics classes in Russian elementary schools and the lower grades of the so-called "basic schools" (grades 5 and 6) include section on *visual geometry*; in other words, students are exposed to what might be characterized as the informal study of geometry.

The aims and objectives of such a program in geometry have by no means always been envisioned in the same way, and their implementation has also varied, so it would be a mistake to suppose that the history of teaching geometry in Russia is the history of a kind of stagnation. On the contrary, the teaching of geometry has been and remains the subject of passionate debate. The authors of this chapter cannot consider themselves neutral with respect to these debates. For example, one of them (A. Werner) had occasion to collaborate over many years with the outstanding Russian geometer A. D. Alexandrov, initially as a participant in his research seminar, and subsequently as the coauthor of his textbooks for schools. It should therefore be stated from the outset that Alexandrov's views on geometry in general and on school geometry in particular are particularly close to him. However, we will attempt to represent other views and approaches that have existed over the past 50 years in Russian schools as well. Since our account will necessarily be limited by the size of this chapter, many mathematical and methodological details will be skipped. On the whole we will focus mainly on the analysis of textbooks and programs, which classroom practices in fact follow in many respects, although it is impossible to describe all the actual and possible varieties of classroom practices here.

2 The Contents of the Course in Geometry in Russian Schools

The contents of the course "Geometry" in the most recent programs at the time of this writing (Standards, 2009) consist of the following sections (the number of hours recommended by the program for the study of each section is indicated in parentheses):

Grades 5 and 6: Visual geometry (45 hours). Students are given a visual sense of basic two-dimensional figures, their construction, and various ways in which they may be positioned with respect to one another, as well as measurements of lengths, angles, and areas. The concept of the congruence of figures and certain transformations of the plane (symmetries) are discussed. Students are also familiarized with three-dimensional figures, their representations, cross-sections, and unfoldings, as well as with formulas for determining their volumes.

Grades 7–9 are devoted to the systematic study of plane geometry, which includes the following sections:

- Straight lines and angles (20 hours);
- Triangles (65 hours);
- Quadrilaterals (20 hours);
- Polygons (10 hours);
- The circle and the disk (20 hours);
- Geometric transformations (10 hours);
- Compass and straight-edge constructions (5 hours);
- Measuring geometric magnitudes (25 hours);
- Coordinates (10 hours);
- Vectors (10 hours);
- Extra time — 20 hours.

In *grades 10 and 11*, geometry is studied at the *basic* and *advanced* levels. Second-generation standards for the upper grades are still being developed, while according to Standards (2004a), at the basic level, students in grades 10 and 11 were required to study the following topics in three-dimensional geometry:

- Straight lines and planes in space;
- Polyhedra;
- Objects and surfaces of rotation;
- The volumes of objects and the areas of their surfaces;
- Coordinates and vectors.

The content of each section is quite rich. For each topic, the programs indicate the basic skill set that the students must acquire. For example, in the section on "Triangles," the students must learn to:

- Identify on a geometric drawing, formulate definitions of, and draw the following: right, acute, obtuse, isosceles, and equilateral triangles; the altitude, the median, the bisector, and the midpoint connector of a triangle;
- Formulate the definition of congruent triangles; formulate and prove theorems on sufficient conditions for triangles to be congruent;
- Explain and illustrate the triangle inequality;
- Formulate and prove theorems on the properties and indications of isosceles triangles, the relations between the sides and angles of a triangle, the sum of the angles of a triangle, the exterior angles of a triangle, and the midpoint connector of a triangle;
- Formulate the definition of similar triangles;
- Formulate and prove theorems on sufficient conditions for triangles to be similar, and Thales' theorem;
- Formulate definitions of and illustrate the concepts of the sine, cosine, tangent, and cotangent of the acute angle of a right triangle; derive formulas expressing trigonometric functions as ratios of the lengths of the sides of a right triangle; formulate and prove the Pythagorean theorem;
- Formulate the definitions of the sine, cosine, tangent, and cotangent of angles from 0° to 180°; derive formulas expressing the functions of angles from 0° to 180° through the functions of acute angles; formulate and explain the basic trigonometric identity; given a trigonometric function of an angle, find a specified trigonometric function of that angle; formulate and prove the law of sines and the law of cosines;
- Formulate and prove theorems on the points of intersection of perpendicular bisectors, bisectors, medians, altitudes, or their extensions;
- Investigate the properties of a triangle using computer programs;
- Solve problems involving proofs, computations, and geometric constructions by using the properties of triangles and the relations

> between them as well as the methods for constructing proofs that have been studied (Standards, 2009, pp. 36–37).[2]

It should be noted that although algebra and geometry are taught as two separate subjects, the course in algebra addresses some topics (concepts) that pertain to the course in geometry as well. One example is the section of the algebra course that covers "Cartesian Coordinates in the Plane"; another is the section on "Logic and Sets" (10 hours) in the second-generation Standards (Standards, 2009, p. 16), which belongs to both the course in algebra and the course in geometry.

Comparing the recently published second-generation Standards for basic schools (cited above) with previously published Standards (Standards, 2004b) or even earlier programs, we find few differences. The contents of the course, in terms of the list of concepts and propositions covered, have remained stable. Naturally, 30 years ago there was no investigation of the properties of a triangle with the help of a computer program, mentioned above, nor was such a problem even posed at the time (nor is it often encountered today in actual classrooms, by all appearances); but problems involving proofs, computations, and constructions that require knowledge of the many theorems studied in the course are assigned and solved today largely as they were years ago.

3 The Aims and Characteristics of the Course in Geometry in Russia

"Why study geometry?" is a question that has been discussed extensively by the international community of mathematics educators, and many arguments have been made in favor of studying geometry (see, for example, González and Herbst, 2006). Russia's official state program in mathematics proclaims the following:

> The contents of the section "Geometry" is aimed at developing students' spatial imagination and logical reasoning skills through the

[2]This and subsequent translations from Russian are by Alexander Karp.

> systematic study of the properties of geometric shapes in the plane and in space and through the use of these properties in solving problems of a computational and constructional nature. A substantial role is also assigned to the development of geometric intuition. The combination of visual demonstrability and rigor constitutes an integral part of geometric knowledge. The sections on "Coordinates" and "Vectors" contain material that is largely interdisciplinary in nature and finds application in various branches of mathematics as well as related subjects. (Standards, 2009, p. 7)

Thus, the teaching of geometry is seen to be of great benefit precisely because of the role that it plays in students' development. Geometry is undoubtedly useful as an applied discipline as well, as is indicated by the conclusion of the quoted passage: natural scientists speak a geometric language, and by failing to teach students this language we compromise their comprehension of the natural sciences and thereby also condemn them to a sort of second-class status in the modern world (whatever the rhetoric employed to legitimize this fact). Russian pedagogy, however, has traditionally harbored the conviction that education is valuable not only and not principally because it conveys various kinds of skills and knowledge that may be subsequently applied directly in practical life, but also because it facilitates the development of students' reasoning skills [this tradition found expression in the works of Vygotsky (1986), which in turn became very influential].

So what is behind this general proposition concerning the development of logical reasoning skills and why is geometry particularly important in this respect? The tradition of major scientists being involved in the writing of courses in geometry, which goes back to Euclid and Legendre, was continued in Russia (USSR), where many outstanding research mathematicians thought about school-level education, wrote school-level textbooks, and, by doing so, have left us their notions about the role and significance of geometry.

In his programmatic article "On Geometry," A. D. Alexandrov (1980) wrote:

> The logic of geometry consists not only in separate formulations and proofs, but in the entire system of formulations and proofs considered

> as a whole. The meaning of every definition, every theorem, every proof, is defined in the final analysis only by this system, which is what makes geometry a unified theory and not a collection of isolated definitions and propositions. This idea of an exact science with a rigorously unfolding system of deductive conclusions, which geometry conveys, is as important as the precision of each conclusion considered on its own. (p. 59)

In other words, geometry teaches students how to analyze and comprehend a system of propositions — how to correlate separate facts, how to look for connections and mutual influences between them. Genuine understanding is possible only through an understanding of the system as a whole. Conversely, although thinking in a fragmentary fashion and ignoring various facts do not entirely preclude all kinds of reasoning, such an approach inevitably makes reasoning more primitive. It would be misleading, of course, to claim that only the study of geometry can teach students a system-oriented approach, but the historic role of geometry as the model for a systematic program (see, for example, Spinoza, 1997) suggests that it would be wise to consider, before rejecting geometry altogether, the possible substitutes that might be found for it in this particular respect within the school program (if any such substitutes exist). We should point out that a comparably *systematic* course in algebra or the natural sciences is likely impossible at the school level (at least we know of no large-scale experiment with any course of this nature).

Another outstanding Russian geometer, A. V. Pogorelov (1974), wrote in the introduction to one of his courses in Euclidean geometry:

> In offering the present course, our basic assumption has been that the main purpose of teaching geometry in school is to teach students to reason logically, to support their assertions with arguments, to prove. Very few of those who graduate from school will become mathematicians, let alone geometers. There will be those who, in their professional lives, will never once make use of the Pythagorean theorem. However, it is unlikely that we would find anyone who will not have to reason, analyze, prove. (p. 7)

At the same time, the logical aspect of geometry stands in a complicated relationship to its visual aspect (as is indicated in the passage from the Standards quoted above). As A. D. Alexandrov wrote:

> The distinctive feature of geometry, which distinguishes it from other branches of mathematics and from all sciences in general, consists precisely in the indissoluble organic conjunction of lively imagination and rigorous logic. Geometry in its essence *is* spatial imagination, permeated and organized by rigorous logic. In any genuinely geometric sentence, be it an axiom, a theorem, or a definition, these two elements of geometry are inseparably present: the visual picture and the rigorous formulation, the rigorous logical deduction. Where either of these sides is absent, there is no genuine geometry. (Alexandrov, Werner, Ryzhik, 1981, p. 6)

The student is in a sense invited to retrace the footsteps of the ancients, who were able to pass from observation to interpretation and abstraction. This experience of systematic mathematical modeling also renders geometry particularly important in the eyes of Russian mathematics educators.

Visual ideas, even visual ideas that are not subsequently proven, are naturally very valuable. A. N. Kolmogorov, perhaps the greatest Russian mathematician of the 20th century, criticized the then-standard textbook by N. N. Nikitin (1961) as follows:

> [The textbook] does not sufficiently distinguish between the two levels at which the material is presented: the logical-deductive level and the visual-descriptive level. The combination of these two levels in textbooks for grades 6–8 seems to me unavoidable. In my opinion, the body of geometric facts with which students become acquainted purely through description might be somewhat expanded.

And he went on:

> But this must not obscure the notion of geometry as a deductive science in the minds of the students. This notion must already become quite clear to them as a result of their study of geometry in grades 6–8. This duality of the school course in geometry must be understandable to the students themselves. They must always know what they are proving and on the basis of which assumptions, what they are simply told on faith, and which conclusions they themselves reach on the

> basis of visual arguments without a clear proof. (Kolmogorov, 1966, p. 26)

Alexandrov saw the opportunity frankly to indicate about virtually all propositions examined in school geometry, whether they were accepted as unproven or rigorously grounded, as well as the opportunity for all students to establish the truth for themselves, without trusting to the authority of a teacher or a textbook — as the enormous potential benefit that geometry had to offer for developing students' minds and worldviews. (Indeed, it is impossible to deny that in other school subjects students must constantly or at least very often trust cited facts, while in geometry classes they become convinced of everything or almost everything on their own.) As Alexandrov (1980) wrote:

> The deep objective of the course in geometry consists of the assimilation of the scientific worldview, of the formation of its foundations. It is shaped by an unequivocal respect for established truth, the need to prove that which is put forward as truth, the refusal to substitute faith or references to authoritative sources for proof. The striving for truth, the search for a proof (or a refutation) — this is the active, and therefore the dominant, aspect of the foundation of the scientific worldview ....
>
> The respect for truth and the demand for proofs convey an extremely important ethical message. In its simplest but very important form, it consists of the imperative not to judge without proving, not to succumb to impressions, moods, and slander where it is necessary to get to the bottom of the facts. Scientific commitment to truth consists precisely of the striving to justify one's convictions about any issue with observations and conclusions that are as objective, as unsusceptible to subjective influences and passions, as is humanly possible. (p. 60)

Below, we will focus on differences between conceptions of the role of geometry and approaches to its teaching; here, we have addressed that side of geometry about which there may be said to be a consensus. Naturally, such complex issues as "the scientific worldview" are almost never mentioned in geometry classes. What an ordinary lesson looks like to working teachers may be imagined, for example, by looking at the methodological recommendations put forward by Glazkov, Nekrasov, and Yudina (1991, or later editions). Let us examine a single eighth-grade class devoted to the rhombus:

> At the beginning of the lesson, the class is asked to solve the following two problems on the basis of drawings that have been made on the blackboard beforehand:
>
> 1. Find the length of two congruent sides of an isosceles triangle whose height is equal to 6 cm and whose vertex angle is equal to 120°.
> 2. The diagonals of a parallelogram are mutually perpendicular. Prove that all of its sides are congruent.
>
> It is then suggested that the teacher formulate a definition of the rhombus and ask the students themselves to define those properties of the rhombus which derive from a definition of the rhombus as a special type of parallelogram, and then to prove specific properties of the rhombus on their own. The recommendations do not stipulate who is to formulate these properties: this may depend on the class; in one class, the students may do this independently, such as using drawings, while in another class it may be done by the teacher.
>
> Thereafter, it is suggested that the students begin solving problems, and it is recommended that the following problems from the textbook be used for this purpose:
>
> - In a rhombus, one of the diagonals is congruent to a side. Find the angles of the rhombus.
> - Prove that a parallelogram is a rhombus if one of its diagonals is an angle bisector.
>
> At the conclusion of the lesson, it is recommended that the students be asked to read on their own the paragraph about squares in the textbook and then to answer the following questions orally, but possibly making use of suggestive drawings prepared by the teacher beforehand:
>
> Is a quadrilateral a square if its diagonals are:
>
> (a) congruent and mutually perpendicular?
> (b) mutually perpendicular and have a common midpoint?
> (c) congruent, mutually perpendicular, and have a common midpoint?

As can be seen, all of the problems are quite traditional. At the same time, it is impossible not to notice that the lesson presupposes active and varied involvement by the students — who, on their own, carry out proofs, construct arguments orally and in writing, and interpret and analyze diagrams. Students are expected to possess a comparatively high level of knowledge about the topics that have already been covered; in order to solve the very first problem, students must know the properties of an isosceles triangle and the relations in a right triangle with a 30° angle. In general, the lesson is conducted as a sequence of problem-solving activities that are connected with one another; for example, solving the problems with which the lesson begins helps to solve the problems that are posed later on, which, therefore, would not be as difficult for the students.

The ability to construct lessons in which intensive reasoning and investigative work will fall within the students' powers is essential for realizing those aims and objectives of the geometry course which we have discussed above and which may be achieved only through systematic and consistent work over many years. At the same time, the stability of the contents of the course also helps teachers to accumulate the necessary teaching experience.

Equally important is that over literally centuries of geometry instruction, an exceptionally rich array of problems and educational and developmental activities has been accumulated. An enormous number of the problems analyzed by Polya (1973, 1981, 1954) were problems in geometry. And this is no accident: to those who want to know "how to solve it," geometry offers special possibilities. Those who believe that students transfer what they have learned — and that by learning to solve problems in geometry students also learn something beyond geometry — cannot afford to turn their backs on geometry. That is why Russian educators do not give up traditional Euclidean geometry.

4 On the Conditions Under Which Geometry Is Taught

The teaching of geometry does not take place in a vacuum. Without setting ourselves the task of listing all of the factors that influence it, we will nonetheless name some of them. To begin with, a great deal is determined by the policies of the Ministry of Education. From the 1930s on, a single geometry textbook was used in all schools in the country (the same was true of all other subjects). Following what was effectively a standoff between the Ministry of Education of the USSR and the Ministry of Education of Russia (Abramov, 2010) during the 1980s, the single textbook was replaced with several different textbooks. With the collapse of the USSR, it was officially proclaimed that any textbook that had been approved by the Ministry of Education could be used for instruction, which formally opened the door to diversity. In practice, however, the process through which textbooks were approved was never completely straightforward and its rules were never entirely transparent (suffice it to say that this process was already skewed simply because the committee that oversaw it met in Moscow, which meant that an overwhelming majority of its members were usually Muscovites). In recent years, the procedure has become even more complicated. It should be borne in mind that general materials for programs are very often developed and approved by the Ministry not before textbooks are written but on the basis of some existing textbook, which thus ends up occupying a privileged position.

On the other hand, a school subject today requires more than just a textbook: it requires an instructional package, which in addition to a textbook includes teaching materials (a set of quizzes and tests to supplement the textbook), a methodology manual for the teacher, and workbooks, which have recently become widespread as well. Of course, the creation of such a package requires a certain amount of support. Whole departments of pedagogical scientific research institutes have worked on the creation of some textbooks, while other textbooks have been developed exclusively by groups of teaching enthusiasts. Today, the creation of new textbooks is sometimes partly sponsored by publishing houses, although the role of publishing houses in Russia to this day cannot be compared with the role that they play in the West. In Russia, authors' enthusiasm and reliance on future success have continued so far to play a primary role (although various grants and direct subsidies from the government are also important).

With respect to general economic issues, it must be pointed out that the significant deterioration of the economic position of teachers during the 1990s, as well as the significant cuts in financing teachers' professional development which took place then and continue to take place now, had a negative impact on the teaching of all subjects, including geometry. Financial problems largely limit the use of computer technologies in geometry classes; even in major cities, schools are usually insufficiently equipped with computers.

Returning to purely methodological issues, however, we should say that the reduction in the number of hours allotted for the teaching of mathematics, which took place over the course of several decades in connection with certain changes in end-result requirements, has resulted in much less time in geometry classes being devoted to the discussion of theoretical questions, i.e. to the students' reproduction of proofs which they have studied, followed by analysis and criticism of these proofs. The role of oral exams in geometry has become less and less important in recent years; with the introduction of the Uniform State Examination and an analogous form of official testing in ninth grade, oral exams have in fact come under the threat of annihilation. This, of course, has had an effect on the orientation of the course in geometry, in which proficiency in oral reasoning is no longer as significant as it once was.

For many years, the teaching of geometry in schools was significantly influenced by college entrance exams (college admissions were based on exams conducted by every educational institution). Analyzing *The Problem Book in Mathematics for College Applicants*, edited by M. I. Skanavi (1988), we can form an idea about the demands that such exams placed on the students. Here, for example, is the first problem from Section A (the easiest of three) in three-dimensional geometry:

> The base of a pyramid is a right triangle with a hypotenuse c and an acute angle of 30°. The pyramid's side edges are inclined toward the plane of its base at a 45° angle. Find the volume of the pyramid (p. 191)

The problems given in entrance exams could (and even should) be criticized for their artificiality or uniformly computational character, but it is evident that exam requirements (and entrance exams to technical colleges have usually included problems in both plane and three-dimensional geometry) have exerted a considerable influence on the attitude toward geometry in schools. College requirements have supplemented and developed the requirements of the Ministry of Education and have been to a sufficiently large degree independent of the latter. The replacement of entrance exams with a uniform state exam means that this independence is coming to an end and that uniformity is being established, the likes of which were not seen even in the days of the Soviet Union.

It should be pointed out here that already in the 1960s it was officially recognized that students differed from one another with respect to their mathematical aptitude and interest in the subject, and schools with an advanced course of study in mathematics appeared in the USSR. Geometry, along with other mathematical subjects, was taught in these schools in an expanded and deeper fashion. In the early 1990s, on the other hand, various kinds of schools with advanced courses in the humanities began to appear, in which students were given an abridged course in mathematics (including geometry). In this chapter, we have no room to discuss the distinctive characteristics of the courses in geometry that we have just mentioned — neither the advanced course nor the abridged one — and our attention will be focused on "ordinary" schools. Nonetheless, the appearance of "not ordinary" schools and classes had an impact on the ordinary course in mathematics. More difficult problems or additional sections, tested out in classes with an advanced course in mathematics, not infrequently found their way into ordinary textbooks as well, even if an asterisk was placed next to them to suggest that they were optional. On the other hand, illustrations or stories that initially appeared in mathematics textbooks for schools with an advanced course of study in the humanities would subsequently migrate to ordinary textbooks without any difficulty at all; showing students something beautiful or entertaining turned out to be natural not only with students who were uninterested in the subject, but with students in general.

Finally, let us mention what is perhaps the most important fact of all. The preparation of mathematics teachers includes serious preparation in geometry over many years. Future teachers come to pedagogical colleges from schools where they studied practically the same deductive course in geometry that they would have to teach. At their pedagogical colleges, they become acquainted with the foundations of geometry, higher-dimensional Euclidean geometry, non-Euclidean geometry, and differential geometry. In addition, they are usually offered various courses in solving "school problems," i.e. problems in elementary plane and three-dimensional geometry. It is naive, of course, to equate the number of courses that students have taken with their actual knowledge, yet it is important to note that considerable time is devoted to geometry in the college program as well. Once again, it must be recalled that at a certain stage, for economic reasons, Russian schools were flooded with out-of-work engineers, whose higher education contained much fewer courses in geometry. We have already pointed out that the system of professional development has been significantly weakened in recent years. Nonetheless, there are still many teachers in Russian schools who are sufficiently well-prepared to carry out instruction in a substantive course in geometry.

5 Toward a History of the Course in Geometry in Russia (USSR)

Below, we will briefly describe the changes that the school course in geometry underwent over the past half-century, without attempting to provide a detailed account of the entire contents of the course (apart from differences that will be specifically mentioned, the course in geometry during the period in question has always been quite similar to the course that exists today, as described above).

5.1 *From Kiselev to Kolmogorov*

Until the mid-1970s, the teaching of geometry in Russian schools was largely based on the textbooks of Andrey Kiselev (1852–1940). The first edition of Kiselev's *Elementary Geometry* came out in 1892 (seven years before Hilbert's *Foundations of Geometry*!), with the following notice on its title page: "For secondary educational institutions" (i.e. for gymnasia and real schools). Before the Revolution, the book gradually conquered the market. Rejected along with the entire old school system during the first post-Revolution years, it made a triumphant comeback in schools during the 1930s (in a somewhat revised version) to become the only geometry textbook used in the Soviet Union. Kiselev's textbook was reprinted even after it ceased to be a recommended school textbook (Kiselev and Rybkin, 1995) and it would be no mistake to say that, to this day, it has been considered by many to be the embodiment of the "good old days," when everything in the schools was supposedly fine.

Kiselev's textbook achieved its popularity for a reason. Written with a knowledge of foreign (above all, French) publications, it grew out of practical teaching experience — first and foremost the experience of Kiselev himself, who spent many years working in secondary educational institutions. Later, I. K. Andronov wrote that Kiselev "knew his strengths and did not undertake to do more than he could do" (Karp, 2002, p. 9). The textbook was rigorous and formally deductive in character, but only to the degree that was accessible to the students of Kiselev's time.

For example, in the first sections on plane geometry, Kiselev freely made use of visual arguments, and his proofs were also formulated using "physical" language; thus, he would refer to figures being superimposed on each other and so on. There is a story dating back to the years after the Second World War (Boltyansky and Yaglom, 1965) about a schoolboy taught, naturally, using Kiselev's textbook — who failed to solve a problem during a mathematics Olympiad because, as he himself wrote, he was unable to prove that a straight line cannot intersect all three sides of a triangle at interior points. The fact that this eighth grader thought about such questions attests, of course, to his exceptional giftedness: questions of this kind, which are certainly quite appropriate for a course in the foundations of geometry, were never raised in Kiselev's textbook at all. What Kiselev proved, generally speaking, was what an ordinary student at a gymnasium or a real school would have found natural to prove.

Kiselev's textbook was comprehensive and logical. Gaps in logic could be found in it, but they were not noticeable to secondary school students (and usually neither to their teachers). The textbook included topics of a general logical nature as well, acquainting students with the notion of the direct theorem, the converse, and the contrapositive.

The course was well structured, and most of the sections into which the textbook was divided could be easily covered in one lesson.

One clearly identifiable strand in Kiselev's course pertains to the *geometry of constructions.* Solving a construction problem involves *analyzing* the conditions of the problem, and it is during this step that the problem's solution is planned; *working out a construction* (i.e. creating an algorithm); *proving* that the figure constructed is in fact the one asked for; and, finally, *investigating* what kind of data are required to solve the problem and how many solutions the problem has. Kiselev's course in plane geometry contains practically no strand that pertains to the *geometry of computations*, but for many years N. A. Rybkin's problem book was used in schools as a supplement to Kiselev's textbook, successfully complementing it.

It must be said, finally, that the dozens of editions that Kiselev's textbook went through permitted its author to continue improving both its scientific and its methodological side.

When Kiselev's textbook first arrived in Soviet schools, it was assumed that it would soon be replaced by a new Soviet textbook, which would take modern trends into account. This, however, did not happen at that time. Over the years that followed, the textbook was increasingly criticized and the need to replace it gradually came to be recognized. Among the criticisms directed against it, the following may be singled out:

- Kiselev's geometry textbooks contained very difficult sections (above all, the chapter on "Similarity"), which, with the introduction of mandatory universal eight-year education (a goal set in the USSR at the end of the 1950s), were beyond the powers of most students.
- Kiselev's course was completely cut off from reality, from practical applications of geometry, which clashed with the policy of the "polytechnization" of education that was being implemented in the Soviet Union during those years. Moreover, it contained no interdisciplinary connections with other school subjects.
- Kiselev's course failed to address many ideas and methods of contemporary, mid-20th-century geometry. It made no mention

of vectors or coordinates; it made almost no mention of transformations. The use of the limit was the only idea that it borrowed from contemporary mathematics.

- The division of the five-year course in geometry into a three-year course in plane geometry and a two-year course in three-dimensional geometry produced the result that, over the three years of studying only plane figures, most students lost their notions of spatial figures, and to revive these at the beginning of the course in three-dimensional geometry would be very difficult. Those students who did not complete a full secondary school course were exposed to no three-dimensional geometric concepts in their geometry course at all.
- Finally, Kiselev's textbook failed to meet several purely curricular needs. For example, in those years, the previously existing separate course in trigonometry was abolished in the USSR, and trigonometry had to be represented more fully in geometry textbooks than it was in Kiselev's textbook.

In 1956, Kiselev's plane geometry textbook was replaced with a textbook by N. N. Nikitin and A. I. Fetisov, which was then itself almost immediately replaced with Nikitin's (1961) textbook *Geometry 6–8*. This textbook, which was very similar to Kiselev's, contained a number of important changes. In particular, the measurement of segments, one of the most difficult topics in Kiselev's textbooks, was substantially simplified — Nikitin presented this topic on a purely visual and intuitive level. The topic "Area" was covered by Kiselev at the end of the course; in Nikitin's textbook, it was shifted to the middle. Finally, in addition to providing a systematic course in plane geometry, Nikitin's textbook presented information, on a visual–intuitive level, about the most important three-dimensional geometric objects — prisms, cylinders, pyramids, cones, spheres — and about the volumes and areas of the surfaces of geometric objects. As a program of study for ordinary, eight-year schools, the course in geometry was now well-rounded and complete. This fact had social significance.

Nikitin's textbook was actively criticized. Kolmogorov (1966) published a long article detailing its shortcomings in the journal *Matematika v shkole*. Perhaps it would have been possible to eliminate these shortcomings in subsequent editions, but it was assumed that there would be only one textbook in the country. In the meantime, textbooks prepared under Kolmogorov's supervision began to appear, and replaced both Nikitin's plane geometry and Kiselev's three-dimensional geometry textbooks.

5.2 *Kolmogorov's Textbooks for Basic Schools*

A general description of the Kolmogorov reforms is given in another chapter of this two-volume set (Abramov, 2010). Kolmogorov himself and the subject committee of which he was the chair devoted great attention to the teaching of geometry. Criticizing existing programs for being outdated, Kolmogorov emphasized that this was especially true of geometry (Kolmogorov, 1967). He envisioned the restructuring of the course in geometry as follows:

> The basic objectives of restructuring the school course in geometry, which have now won the widest acceptance, may be formulated in terms of three propositions:
>
> 1. The formation of elementary geometric concepts should take place in the first years of school.
> 2. The logical structure of the systematic course in geometry in the middle grades should be substantially simplified by comparison with the Euclidean tradition. At this stage, students should become habituated to rigorous logical proofs while the right to accept a redundant system of assumptions without proof should also be openly recognized.
> 3. The course in geometry in the higher grades should be founded on vectorial concepts. In this respect, it would also be natural to rely on the coordinate method (but only in an auxiliary fashion, so that the presentation does not become less "geometric" as a result of the reliance on this approach). (Kolmogorov, 1967, p. 11)

Some of these assertions may give rise to objections (for example, it is by no means an established fact that the vector-based approach to geometry instruction is simpler or in any way superior to the traditional approach). What is important, however, is that Kolmogorov envisioned the creation of the new textbook as an open process that would rely — just as the creation of Kiselev's textbook had relied — on international findings. Kolmogorov wrote:

> In order to make it possible to work calmly and confidently on new geometry textbooks, preliminary work must be carried out at once: one or several working groups of scholars and teachers, using foreign findings, must put together and publish the outline (or several outlines) of a "logical skeleton" of a school course in geometry (the basic assumptions and the basic sequence of theorems with proofs) in a form that will be open to criticism and experimental use by sufficiently experienced teachers. (Kolmogorov, 1967, p. 13)

Unfortunately, this was not done.

An idea of some of the aims set by Kolmogorov during the writing of the textbook (which he himself oversaw) is conveyed by the following statement made by him:

> We have decided to retain separate geometry textbooks for grades 6–10. By comparison with a system of unified textbooks in mathematics, which is the norm in many countries, the existence of a separate geometry textbook has some advantages, but only if the logic of the construction of the geometry course is rigorously coordinated with the courses in algebra and elementary analysis. (Kolmogorov, 1971, p. 17)

It was expected that such *rigorous coordination* could be achieved, in part, by organizing the presentation of the material around geometric transformations.

The new course in geometry was structured on the basis of set theory. This led to the appearance in schools of the term "congruence," which became perhaps the most frequently mentioned example of the difficulty of Kolmogorov's course — prior to it, as well as afterward, people spoke about the "equality" of figures. Since in Kolmogorov's course figures were seen as sets of points, and a set was "equal" only to itself, it was impossible, in the opinion of Kolmogorov and his coauthors, to talk about "equal triangles," as had been done before (Kolmogorov *et al.*, 1979). Triangles that could be superimposed through a geometric transformation that preserved distances (rigid motion) began to be characterized as "congruent." It seems unlikely that the introduction of one new term by itself could have exceeded students' capacities sufficiently to warrant discussions about their suffering (which were not unusual for the pedagogical periodicals of the time and indeed are not unusual today). On the other hand, the introduction of a new term always creates certain difficulties, and if it could have been avoided, for example, by specifying the precise meaning that was being ascribed to the old term, then fighting so hard for the new term, and turning it into a rallying cry, hardly seems worthwhile.

What probably happened to be more important was that many proofs turned out to be fundamentally new and unfamiliar. For example, Kiselev and his followers had proven the classic theorem that the diagonals of a parallelogram $ABCD$ bisect each other (Fig. 1) by examining the triangles AOD and BOC (O is the point of intersection of the diagonals). It is not difficult to see that these triangles are congruent (or "equal," to use the term of that time), from which everything immediately follows.

Kolmogorov's approach was to examine the midpoint O of the diagonal $\overline{BD}$ and point reflection with respect to this point. Since it was stated at the outset that a point reflection maps a straight line to a parallel straight line, and since it is clear that point B, under such reflection, is mapped to point D, while point D is mapped to point B, it was possible to conclude that, under the point reflection being examined, the straight line $\overleftrightarrow{AD}$ is mapped to the straight line $\overleftrightarrow{BC}$ (as the only straight line which passes through point B and is parallel to $\overleftrightarrow{AD}$). In an analogous manner, it was proven that the straight line $\overleftrightarrow{AB}$ is mapped to the straight line $\overleftrightarrow{DC}$. Thus, it was concluded that, under

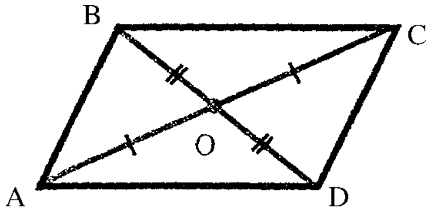

Fig. 1.

the given point reflection, point A is mapped to point C, which proves that O is the midpoint of the diagonal $\overline{AC}$.

Kolmogorov's proofs, which were in their own way beautiful and vivid, were nonetheless often difficult to grasp. In addition, if a student using Kiselev's textbook had the impression that all of the propositions to which reference was made were completely proven (whether this impression was correct or not is another matter), then Kolmogorov's textbook did not foster such an impression, if only because it attempted to set a much higher level of rigor than Kiselev's textbook did.

Discussing the axiomatic approach, Kolmogorov (1968) wrote:

> But in schools it has become common practice merely to indicate "examples of axioms." The actual list of these examples of axioms is usually laughably short. Apparently, the students are never asked to analyze a proof by identifying all of the axioms on which it is based. Meanwhile, such an exercise should be insistently recommended: the proof of theorem T relies on theorems T_1 and T_2, the proof of theorem T_1 relies on axioms A_1 and A_2, while the proof of theorem T_2 relies on axiom A_3 and theorem T_3, and so on, until only axioms remain. (p. 22)

It may be objected, however, that such an exercise is quite difficult for ordinary public school students if they are dealing with a theorem that has any substance. Even more significantly, such an exercise might give rise to a misguided notion of geometry as a subject in which there is a strange ritual of explaining what is obvious at great length for unknown reasons (this is especially the case if, as unfortunately often happens in Western textbooks, the theorem being examined is a very simple one, consisting of one or two steps).

The first chapter of Kolmogorov's textbook *Basic Concepts of Geometry* formulates and enumerates 15 propositions. Nine of them are axioms. Five are proven; one is illustrated. Of the five proofs of the propositions, four are one step away from the axioms on which they are based, and only one (the derivation of a formula for distance between points on a coordinate line) contains more than one logical step. The textbook *Geometry 6* (Kolmogorov, 1972) contains 38 separate propositions in all, over half of which are not proven.

We will not discuss the other methodological innovations that provoked criticism — such as the approach to defining vectors in Kolmogorov's textbook and the accompanying textbook by Klopsky *et al.* 1977 — or, on the contrary, met with success (such as the replacement of a separate problem book with sections on "Questions and Problems" in the textbook itself). Making the course at once more rigorous and more simple, which was Kolmogorov's goal, is not an easy task. Kolmogorov and his coauthors took many revolutionary steps. Possibly, given many years of further work, many difficult spots might have been smoothed over. At least, Kolmogorov (1984) himself later wrote:

> The question of when it is proper to begin talking to students about geometry's logical structure should be discussed again. The experience of working with different versions of geometry textbooks over the past decade has shown that doing so at the beginning of sixth grade is premature. (pp. 52–53)

But no more time was allowed for correcting, rethinking, and revising. A major campaign (Abramov, 2010) effectively resulted in the setting of a new agenda: to create new textbooks with the aim of replacing Kolmogorov's.

5.3 *Geometry Textbooks for Basic Schools from the Late 1970s to the 1980s*

The vicissitudes of the struggle against Kolmogorov's reforms and subsequent events are described in A. M. Abramov's chapter in this two-volume set. The campaign that unfolded at the time went beyond the bounds of a debate about methodology and became politicized, giving rise to a situation in which even observations that were fundamentally correct were exaggerated to the point of becoming nonsensical. Thus, the need to give up set theory, which was allegedly the cause of all difficulties and unsuitable for Soviet children in general, became one of the campaign's slogans. It was decided that the word "set" would not be used. The plan developed by I. M. Vinogradov's committee contained an explicit proposal "not to use set theory as the basis for the teaching of mathematics in secondary schools" (Proekt, 1979, pp. 7–12). A. V. Pogorelov's textbook, which had been written by this time, turned out to be "good" because the rejected word was not used in it. In his article "On the Concept of the Set in the Course in Geometry," A. D. Alexandrov (1984a) showed that the content of this textbook, which had been recommended by Vinogradov's committee, was in fact based on set theory. The same fact had been pointed out even earlier by Kolmogorov (1983), in his memo "On A. V. Pogorelov's Teaching Manual *Geometry 6–10*":

> The very first page of the textbook states: "We conceive of every geometric figure as being composed of points." It is difficult to understand this sentence except as an assertion that every figure is a set of points. However, the actual word "set" is not used anywhere in the textbook. (p. 45)

But, by this time, not using unapproved words was precisely what was important.

A. M. Abramov describes, in his detailed account, how the decision was made to conduct a nationwide competition for mathematics textbooks, in which all of the principal authors' groups took part, and how the defeat of Kolmogorov gave other working groups a chance to make their own proposals heard. In addition to Pogorelov's textbook, which has already been cited, we should also mention the textbook written under the supervision of the academician A. N. Tikhonov by L. S. Atanasyan (1921–1998), chair of the geometry department at the Moscow State Pedagogical Institute; Professor E. G. Poznyak (1923–1993) of the mathematics division of the Moscow State University's physics department; Poznyak's colleagues V. F. Butuzov and S. B. Kadomtsev; and the well-known mathematics educator I. I. Yudina, who joined them later. Another working group, formed under the supervision of the academician A. D. Alexandrov, included one of the authors of this chapter, A. L. Werner, who was then chair of the geometry department at the Leningrad (now St. Petersburg) Pedagogical Institute, and V. I. Ryzhik, a well-known teacher.

These three textbooks, which went on to become probably the most popular, won the competition. The manuscripts of the textbooks written by Atanasyan and his colleagues won first place in both competitions ("Geometry 7–9" and "Geometry 10–11"). Second place in the competition "Geometry 7–9" was won by Pogorelov's textbook, while in the competition "Geometry 10–11" Pogorelov's textbook shared second and third place with the manuscript presented by the Kiev authors G. P. Bevz, V. G. Bevz, and N. G. Vladimirova. The manuscript of the textbook "Geometry 7–9" by Alexandrov, Werner, and Ryzhik came in third, and their "Geometry 10–11" fourth. Below, we describe these textbooks' approaches in greater detail.

5.3.1 *A. V. Pogorelov's geometry textbook*

Long before the nationwide competition, A. V. Pogorelov, an academician and well-known geometer, published a book in elementary geometry (1974), which became the foundation for his school textbook. Therefore, we will begin with his textbook (Pogorelov's textbook was reissued many times; see, for example, Pogorelov, 2004a, 2004b.) The competition committee characterized his work as follows: "The manuscripts of the textbooks are characterized by a high level of rigor in the presentation of the theoretical material, brevity and precision of language, and the use of an axiomatic foundation in the construction of the course" (Konkurs, 1988, p. 49).

What Kolmogorov had been preparing to do (but did not do), Pogorelov did: at the very beginning of the course, he named the basic geometric figures — *point* and *straight line* — and presented a complete system of axioms for this course, which he described as the fundamental properties of the basic geometric figures. After this, precisely and methodically, Pogorelov presented definitions and proved subsequent propositions. The course is unified, self-contained, and similar to a course in the foundations of geometry.

Pogorelov's geometry textbook is structured as an outline. It is divided into sections which are broken down into clauses. The theoretical text in each section is followed first by test questions and then by problems. People who worked with Pogorelov told the authors of this chapter that he always strove to shorten the text of his textbook and would repeat: "If you see that a sentence can be crossed out, then cross it out!" Pogorelov assumed that teachers by themselves would add the necessary words in class, in accordance with their pedagogical approach.

The hand of an outstanding geometer can be seen in many of the proofs and in how the presentation of the topics is structured. Nonetheless, as the textbook was put into use, critical observations arose. Let us return, for example, to the theorem about the intersection of the diagonals of a parallelogram, discussed above. Kiselev's tacit introduction of a point O at the intersection of the diagonals was unacceptable for Pogorelov's course, which was far more rigorous than Kiselev's: indeed, it does not follow from anything that a parallelogram's diagonals intersect at all. Kolmogorov's proof, examined above, showed this, but it relied on transformations, which was unacceptable for Pogorelov's course. The way out of this predicament was found, first, by proving on the basis of the congruence of the triangles (which was once again referred to as "equality") that if the diagonals of a quadrilateral intersect and their point of intersection divides them in half, then this quadrilateral is a parallelogram. As for the theorem that the diagonals of a parallelogram intersect and are divided in half by their point of intersection, it was proven as follows (Fig. 2):

> In the parallelogram $ABCD$, consider the midpoint O of the diagonal $\overline{BD}$, draw the segment $\overline{AO}$, and extend it to a point C_1, such that the length of $\overline{AO}$ equals the length of $\overline{OC_1}$. The quadrilateral ABC_1D turns out to be a parallelogram in accordance with a theorem proven earlier. From this it follows that the straight line $\overleftrightarrow{DC_1}$ is parallel to the straight line $\overleftrightarrow{AB}$; therefore, $\overleftrightarrow{DC_1}$ coincides with the straight line $\overleftrightarrow{DC}$ (since, given a point and a straight line, there is only one straight line that passes through the point and is parallel to the given

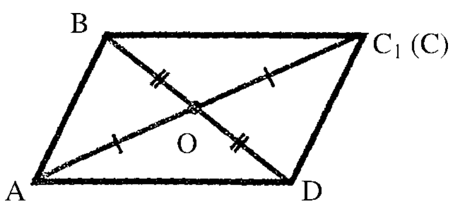

Fig. 2.

> straight line). In an analogous manner, it is proven that the straight line $\overleftrightarrow{BC_1}$ coincides with the straight line $\overleftrightarrow{BC}$ as well. As a result, we find that the points C_1 and C are identical, which means that the parallelograms $ABCD$ and ABC_1D are identical, and therefore in the given parallelogram $ABCD$ the diagonals intersect and their point of intersection divides them in half.

For the students, this proof was decidedly not simple. In some cases, the efforts made to formulate a precise proof of the fact that some pair of straight lines intersected — a fact that was visually obvious to the students — completely overshadowed the substantive part of the theorem in the students' eyes. The brevity of the textbook, which was meant to offer teachers an opportunity to make their own contributions, often simply made the lessons bare and dull if the teachers had nothing to add.

Pogorelov's textbook has remained in use to this day, although it appears to be substantially less widespread now than in the 1980s.

5.3.2 *The geometry textbooks of L. S. Atanasyan and his coauthors*

L. S. Atanasyan and his coauthors began working on their textbooks over 30 years ago, in late 1970s, and today these textbooks are the most popular in Russia (Atanasyan *et al.*, 2004, 2006). In its day, the nationwide competition committee characterized them as follows: "The manuscripts are distinguished by the fact that the presentation of the material in them is accessible, by the fact that they are oriented toward students studying the material on their own, and by their explicit practical orientation" (Konkurs, 1988, p. 49).

In a conversation with one of the authors of this chapter (A. Werner), E. G. Poznyak himself said that the aim of their working group was to develop a simple textbook in the spirit of Kiselev's. Indeed, Atanasyan and his coauthors returned to the reliable path of Euclid (and Kiselev), which has stood the test of millennia. For example, they call two geometric figures "equal" if they coincide when superimposed on each other. This is exactly what we find in Kiselev and almost exactly what we find in Euclid. The proofs of the congruence of triangles, and of much else, are what they are in Kiselev and even what they are in Euclid. The theorem about the intersection point of two diagonals, which we examined above as an example, was once again reunited with its old proof from Kiselev's textbook, which relied on the congruence of triangles and never raised the question of whether the diagonals intersected at all. All of this was known and familiar to teachers, to students, and to students' parents.

Certain innovations appeared in the discussion of similarity. Kiselev, following French models (Barbin, 2009), had departed from the Euclidean principle of using areas to prove theorems about relations between the lengths of segments. As a consequence, the theorems on which the basic propositions about similar triangles relied turned out to be very difficult: Pogorelov had made one such theorem a required part of his course (in his formulation, it read as follows: *the cosine of an angle depends only on the angle's degree measure*), but in practice it turned out that students did not understand it. The textbook of Atanasyan *et al.* (just like the textbook of Alexandrov and his coauthors, which will be discussed below) returns to the spirit, if not the letter, of Euclid's approach, using areas to prove theorems about similarity. This noticeably simplified the course, not to mention the fact that introducing the concept of area early on made discussions of many geometric ideas and problems more accessible earlier than they had been previously.

The chapters devoted to post-Euclidean geometry are arguably more open to criticism. For example, according to what we have observed, the concluding chapter of the course in plane geometry, "Rigid Motion," is almost never studied in school in practice (the key section concerning the relationship between the concept of rigid motion, introduced in this chapter, and the concept of congruence, examined earlier, is marked with an asterisk, which denotes that material in the section is optional). Moreover, the idea of discussing transformations of the plane after all else in the course has been covered might itself give rise to objections.

On the other hand, the range of problems offered in the textbook of Atanasyan and his coauthors is rich and convenient for teachers. These problems, along with good methodological supporting materials (teachers' manuals), appear to have been one of the most important reasons for the success of this textbook. Every section is accompanied by problems. Often, two similar problems are given in a row: one of them is solved by the teacher in class, and the other is assigned as homework. Each chapter also contains additional problems, and at the end of each class there is a set of more difficult problems. Questions for review follow each chapter. In addition to problems, practical assignments accompany some sections, when appropriate. Answers to problems and hints for some solutions appear at the end of the textbook.

5.3.3 *The textbooks of A. D. Alexandrov and his coauthors*

The manuscript of the geometry textbook for grades 7–9 by A. D. Alexandrov and his coauthors was characterized by the nationwide competition committee as follows: "It is distinguished by its untraditional treatment of a number of topics, by the liveliness and readability of its language, by the overall orientation of its exercises toward students' development" (Konkurs, 1988, p. 49).

Indeed, if the traditional view was that a geometry textbook should be laconic and dry, then the authors of this textbook (Alexandrov *et al.*, 1983, 1992, 1992, 2006), and above all Alexandrov himself, strove to speak to the teacher and the students in a completely different language, not only explaining various propositions to them but also discussing their content and meaning. Below, for example, is a brief excerpt from the section of the textbook in which Alexandrov explains the meaning of the Pythagorean theorem:

> The Pythagorean theorem is also remarkable because in itself it is not at all obvious. If you look closely, for example, at an isosceles triangle with an added median, then you will be able to see directly all of the properties that are formulated in the theorem that deals with it. But no matter how long you look at a right triangle, you will never see that its sides stand in this simple relation to one another: $a^2 + b^2 = c^2$. Yet this relation, as a relation between the areas corresponding to the sides, becomes obvious from the construction

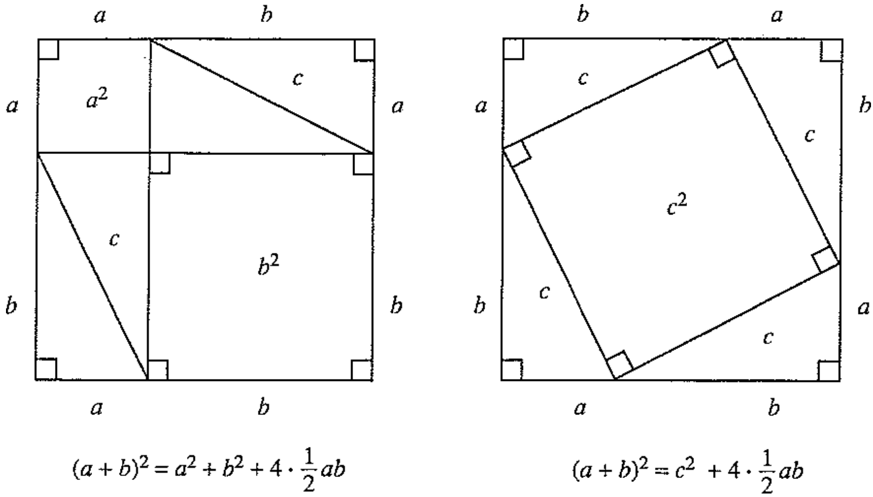

Fig. 3.

depicted in Fig. 3. This is what the best style of mathematics consists of: taking something that is not obvious and making it obvious by means of a clever construction, technique, or argument. (Alexandrov *et al.*, 1992, p. 139)

Alexandrov formulated several principles for teaching geometry (following Werner, 2002, p. 166):

- Since one of the aspects of geometry is its rigorous logical character, and since the students of grades 7–11 are already capable of grasping this logical character, the course in geometry must be presented sufficiently rigorously, without logical gaps in the *basic structure* of the course.
- Since the second basic aspect of geometry is its visual character, in the teaching of geometry every element of the course should be initially presented in the most simple and visually intuitive way, using that which may be drawn on the blackboard, demonstrated on models, on real objects, as far as possible.
- Further, despite its high degree of abstraction, geometry arose from practical applications and is put to practical uses. Therefore, the teaching of geometry must unquestionably connect it with

real objects, with other disciplines, with art, architecture, and so on.

- A textbook aimed at ordinary secondary schools must not contain in its basic part anything that is extraneous, of secondary importance, or of little significance in the main body of the text.
- But since the abilities and interests of the students are quite varied, such a textbook must contain supplementary material, aimed at students who are stronger and have a greater interest in mathematics.
- Geometry must be presented geometrically. It contains its own methodology: the direct geometric methodology of grasping concepts, proving theorems, and solving problems. The synthetic methodology of elementary geometry must not be squashed in school-level instruction by any coordinate-based methodology, vector-based methodology, or any other methodology. The direct geometric methodology is simpler, more important, and more natural for the purposes of a general secondary education and corresponds to the very essence of geometry. It is needed by anyone who deals with three-dimensional objects.
- The school course in geometry *must be connected with contemporary science, must include, as far as this is possible, elements of contemporary mathematics*. In addition, the course in geometry, as a logical system in which everything is proven, is important for developing the rudiments of a scientific worldview, which demands proofs rather than references to authoritative sources.
- But since *there is simply no such thing as absolute rigor, a certain level of rigor must be selected and established*, and this level of rigor must be maintained through the entire course. The course must not have logical gaps, at least in its basic sequence of topics. Otherwise, it will lose its systematic aspect, the logic of the exposition will become blurred, and students will be exposed not to a unified science — geometry — but only to its fragments.

In this way, the three foundations of the textbook, according to Alexandrov's way of thinking, were supposed to be visual explanations, logic, and connections with the real world and practical applications. Such a conception of the author's task led Alexandrov to present many sections in a new way. For example, the fundamental object that he chose was not the straight line, as was the norm in other school textbooks, but the segment, since it was precisely this that people dealt with in practice. For the same reason, the traditional axiom of parallel lines — which states that through a point outside a given straight line, only one straight line may be drawn that is parallel to that line — was replaced by the axiom of the rectangle. This axiom postulates that it is possible to construct a rectangle whose sides are equal to given segments (the possibility of such a construction is confirmed by everyday practice).

Alexandrov used the congruence of segments — visually apparent and "testable" — to define other concepts, including the congruence of figures. Untraditional definitions (although equivalent to traditional ones) were given in the textbook for other concepts as well — for example, the similarity of triangles.

All of this made the presentation shorter and more visual. At the same time, although Alexandrov's approach was based precisely on a deep understanding of the classical tradition, the novelty of many of the ideas scared off some teachers when they suddenly discovered that from now on they would have to teach the congruence (equality) of triangles in a way that differed from what they had been accustomed to for years.

5.4 *Textbooks That Appeared After the Collapse of the USSR*

In the early 1990s, geometry in Russian schools was taught using the textbooks that had won the nationwide competition. But already by the mid-1990s new geometry textbooks began to appear. If in the USSR the commission to write a textbook came from the government, then now it became possible to create a new textbook based on the private initiative of some specialist or some organization that wished to finance such work. If in the USSR only one publishing house, *Prosveschenie*, published instructional materials for schools, then in the 1990s dozens of publishing houses appeared in Russia that were involved in publishing instructional materials and textbooks that competed in a market. Among the new publishing houses were *Drofa*, *MIROS* (which combined publishing activities with scientific research), *Mnemozina*, and *Spetsial'naya literatura*. In the late 1990s, the Russian Ministry of Education, together with the National Training Foundation (NTF), conducted several competitions for "New Generation Textbooks" for different grades (1st–11th) and in different subjects. These competitions touched on the teaching of geometry as well, and certain new textbooks were prepared for them and subsequently published with proper support. Below, we will describe briefly some of the books that appeared during those years. We remind the reader that because in this chapter we are focusing on so-called ordinary schools, we will not discuss textbooks for schools with an advanced course of study in mathematics or schools specializing in the humanities.

5.4.1 *I. F. Sharygin's textbooks*

I. F. Sharygin (1938–2004), a graduate of Moscow State University, accomplished much for geometry education in Russia. From his pen came several problem books which contained diverse and difficult problems (Sharygin, 1982, 1984) and were a crucial source for gifted students interested in geometry. At the same time, in collaboration with L. N. Erganzhieva, he published the book *Visual Geometry* (Sharygin and Erganzhieva, 1995), which, along with the sections on geometry that he wrote for the textbook edited by him and G. V. Dorofeev (Dorofeev and Sharygin, 2002), did much for the geometric development of fifth and sixth graders. Recalling Albert Schweitzer's words concerning reverence for life, one can say that what was characteristic of Sharygin was reverence for geometry — a sense of awe based on a profound knowledge of the properties of geometric figures and, one might say, a personal relationship with these figures. One of the authors of this chapter (A. Karp) remembers how, during one conference, Sharygin with sincere pain spoke about the fact that although school geometry was based on the triangle and the circle, in the geometry course the triangle receives much attention but the circle is forgotten and pushed into the background. Sharygin tried to convey this awe for geometry not only in the books cited above but also in his textbooks (Sharygin, 1997, 1999). The annotations to them state:

> The new textbook in geometry for ordinary schools embodies the author's visual–empirical conception of a school course in geometry. This is expressed first and foremost in the rejection of the axiomatic approach. Axioms, of course, are present, but they are not foregrounded. Greater attention, by comparison with traditional textbooks, is devoted to techniques for solving geometric problems. (Sharygin, 1997, p. 2)

Addressing the students, Sharygin writes:

> Far from all students feel a great love for mathematics. Some are not too good at carrying out arithmetic operations, have a poor grasp of percentages, and in general have reached the conclusion that they have no mathematical abilities. I have good news for them: geometry is not exactly mathematics. At least, it's not the mathematics with which you have had to deal up to now. Geometry is a subject for those who like to daydream, draw, and look at pictures, those who know how to observe, notice, and draw conclusions. (Sharygin, 1997, pp. 3–4)

Sharygin's textbooks are full of illustrations, including the works of M. C. Escher, Victor Vasarely, and Anatoly Fomenko. The mathematical content of his textbooks, however, is quite traditional. In his posthumously published article "Do Twenty-First Century Schools Need Geometry?" Shargyin (2004) identified three basic types of courses that taught anti-geometry (false geometry and pseudogeometry). The first type is built on a formal–logical (axiomatic) foundation; the second type is the practical–applied course with a narrowly pragmatic profile; and about the third type he wrote: "And yet I am convinced that the coordinate method (along with trigonometry) constitutes one of the most effective means for ruining geometry, and even for destroying geometry" (p. 75). Nonetheless, both axioms (basic properties) and trigonometry with coordinates are to a certain degree present in his textbooks as well.

Sharygin introduced into his textbooks sections that were usually not included in textbooks for ordinary schools (he did, however, mark them with an asterisk to indicate that they are optional and not part of the mandatory program). For example, the textbook for grades 10 and 11 discusses the Schwarz boot, and in its chapter on regular polyhedra half of the sections are optional, including a section explaining that the number of regular polyhedra is finite.

5.4.2 *The textbooks of I. M. Smirnova and V. A. Smirnov*

In contrast to Sharygin, the authors of these textbooks (Smirnova and Smirnov, 2001a, 2001b), professors at Moscow Pedagogical University, emphasize their adherence to the axiomatic approach. A note in their textbook *Geometry 7* states:

> The textbook is based on the axiomatic approach to structuring a course in geometry and corresponds to the mathematics program in ordinary schools. In addition to classical plane geometry, topics in spatial geometry, contemporary geometry, and popular-scientific geometry have been included in it as supplementary material. (Smirnova and Smirnov, 2001a, p. 2)

The content of the course is wholly traditional (in particular, the authors once again return to Kiselev's approach to the defining similarity, presenting a theorem about the proportionality of segments cut off by parallel straight lines from the sides of an angle, a theorem that is effectively unprovable in school). This textbook contains fewer problems than, for example, the one by Atanasyan *et al.*

The authors strive to make their geometry textbooks interesting and entertaining. The following words are printed on the covers of the textbooks: "Geometry is not hard. Geometry is beautiful." These textbooks probably contain even more optional sections than the one by Sharygin (1999). Thus, at the end of seventh grade, six optional sections are given: "Parabola," "Ellipse," "Hyperbola," "Graphs," "Euler's Theorem," and "The Four-Color Problem." The textbook *Geometry 10–11* includes such sections as "Semiregular Polyhedra," "Star Polyhedra," "Crystals — Nature's Polyhedra," "The Orientation of Space," "The Moebius Strip," and "Polyhedra in Optimization Problems."

5.4.3 *The textbooks of A. L. Werner and his coauthors*

After Alexandrov's death, his coauthors and collaborators prepared several new textbooks. Adhering to the same principles on which the earlier textbooks had been based, the authors attempted to address certain critical new problems.

One of them consisted in the need to fill out the course in plane geometry with elements of three-dimensional geometry. We have already pointed out that without this, the spatial imagination of students who are immersed for three years in the world of plane geometry grows weaker (or atrophies altogether). The problem of overcoming students' *spatial blindness* is well known to teachers who are beginning to teach a course in three-dimensional geometry. Furthermore, since a complete (11-year) secondary education once again became nonmandatory, it was deemed necessary to provide students with some rudimentary knowledge of three-dimensional geometry in basic schools (the nine-year program).

The authors of existing textbooks often merely supplemented their plane geometry textbooks with one last chapter, which presented the rudiments of three-dimensional geometry. This in no way solved the problem of developing students' spatial imaginations: as before, they were immersed for three years in the world of plane geometry. Therefore, during the very first year of competitions, the NTF announced a competition for a new textbook, *Geometry 7*, and during the second year, for a second textbook, *Geometry 8–9*, in which a systematic course in plane geometry would be supplemented with elements of three-dimensional geometry, presented in a visual–intuitive fashion. Both competitions were won by textbooks written by a working group that included A. L. Werner, V. I. Ryzhik, and T. G. Khodot, an associate professor at Herzen University's geometry department (Werner *et al.*, 1999, 2001a, 2001b).

The elements of three-dimensional geometry in these textbooks were presented along with analogous topics in plane geometry: perpendiculars in a plane were accompanied by perpendiculars in space, parallels in a plane were accompanied by parallels in space, the circle and the disk were accompanied by the sphere and the ball, and so on. Each textbook placed special emphasis on the main theme of each course: in grade 7, the geometry of constructions; in grade 8, the geometry of computations; in grade 9, the ideas and methods of post-Euclidean geometry — vectors, coordinates, and transformations.

The same main themes were followed in another series of textbooks prepared by Werner and Ryzhik on the basis of textbooks prepared under the supervision of Alexandrov as part of a project to create the so-called *Academic School Textbook* (the heads of the project were the academician V. V. Kozlov, vice president of the Russian Academy of Sciences; the academician N. D. Nikandrov, president of the Russian Academy of Education; and A. M. Kondakov, general director of the Prosveschenie publishing house and corresponding member of the Russian Academy of Education).

Among the distinctive features of this series of textbooks (Alexandrov *et al.*, 2008, 2009, 2010) were their sections on logic and set theory, as well as their heightened attention to the history of geometry. The textbooks placed considerable emphasis on issues pertaining to the language of geometry, providing translations of geometric terms accompanied by lists of words with the same roots. They contained numerous illustrations showing various architectural constructions ("frozen geometry") and discussed symmetry and its role in connection with this, and so on.

Ryzhik broke down the problems in the book into sections whose titles indicated to teachers and students the main form of activity involved in solving them. Among these titles were the following:

- *Analyzing solutions.* Students are not only given completed proofs, which are part of the theoretical course, but also shown how these proofs are found.
- *Supplementing theory.* Students are given theoretical propositions that do not belong to the main theme of the course, but are useful for solving other problems. Students can refer to them along with the theoretical propositions that belong to the main theme of the course.
- *Looking.* Students are taught to interpret information presented in visual form, and students' spatial (two- and three-dimensional) imaginations are developed.

- *Drawing*. Students develop their spatial thinking skills.
- *Representing*. The problems in this category may be solved using only visual representations, without boring theoretical explanations.
- *Working with formulas*. Important problems that link the courses in geometry and algebra.
- *Planning*. Designing an algorithm that leads to the solution of a problem.
- *Finding the value*. Ordinary classroom computation problems.
- *Proving*. Problems involving proofs.
- *Investigating*. Problems whose conditions or possible results may contain some uncertainty, incompleteness, and ambiguity.
- *Constructing*. Construction problems.
- *Applying geometry*. Problems from outside mathematics that must be translated into mathematical language.

6 Concerning Some Problems with the Course in Geometry in Russia in Recent Decades

Pondering the development of and changes in the course in geometry in Russia over the past half-century (a short description of which has been given above), one cannot help noticing several basic problems around which discussions have revolved. It must be acknowledged that the existence of these discussions in itself shows that these problems are difficult and that no simple solutions to them can be expected. They can, however, be examined in greater detail, as we will attempt to do below.

6.1 *The Problem of the Rigor of the Course in Geometry*

At the very beginning of this chapter, we discussed the importance of the logic and rigor of the school course in geometry. Obviously, however, the level of rigor found in Hilbert is not the same as that found in Euclid. What level of rigor, then, do schools need? This pertains not only to proofs of propositions but also to the rigor of definitions and to the precision of language in general.

Let us begin with the latter. It is not difficult to see that Kiselev's textbook, which has to this day been considered a model of rigor and deductive logic, contains propositions that turn out to be simply false because of what may be called linguistic sloppiness. For example, it contains the following theorem: "The three altitudes of a triangle intersect at one point" (Kiselev and Rybkin, 1995, p. 108). Meanwhile, generally speaking, an altitude is a perpendicular dropped from a vertex of the triangle to the side opposite it or its extension. Obviously, in an obtuse triangle, the altitudes do not intersect at one point; rather, it is the straight lines that contain the altitudes that intersect at one point. Moreover, generally speaking, from a certain point of view, virtually all theorems that involve areas and volumes are meaningless. Say, consider the statement "The area of a triangle is equal to one half of the product of its base and height." How can one multiply a base, i.e. a segment? One should refer, rather, to the length of the base.

Kolmogorov (1971) wrote: "Traditional geometry textbooks are weighed down by the extreme polysemy of their definitions and notations" (p. 17). It turned out, however, that avoiding such polysemy completely is very difficult, while using symbolic notations overburdens the teaching of the course and, most importantly, alters somewhat its direction. The student in effect has to learn a new language and then to pay attention to subtleties of notation — making sure to distinguish between $\overline{AB}$, $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, and other expressions, instead of focusing on geometry itself. Of course, no one would deny that it would be good if all students acquired a command of precise mathematical symbolic notation, but usually the time that teachers have at their disposal is limited and they must choose what to spend it on. Russian textbooks subsequently simplified symbolic notation, writing simply AB, verbally indicating what was meant or even expecting students to understand what was meant from the context.

Precise definitions are indispensable in mathematics (as in any other science). Moreover, they are vital in everyday life [recall the example cited by Vygotsky (1986) of a child who said that someone had once been the son of some woman but was not her son any longer: the child had formed his definition of "son" spontaneously and associated it with a certain age — thus, an adult could not be a son!]. The problem, however, is that to give a rigorous definition of, say, a polyhedron is very difficult (Alexandrov, 1981); meanwhile, students already have an intuitive notion of it, which is sufficient for solving certain problems, including quite substantive ones. This intuitive notion may be made more precise when necessary, and various relevant details may be mentioned explicitly, which can itself be useful, but striving to give a complete and precise definition of a polyhedron is probably not useful (at least attempts to do so in Russian textbooks have not met with success — teachers and students have usually simply skipped over them). As Alexandrov (1984b) emphasized: "The purpose of definitions is not for students to memorize them by rote, but to make students' understanding more precise. We must try to achieve not empty memorization, but effective learning, i.e. learning that allows students to apply what they have learned" (p. 45).

Consequently, in dealing with any new concept, the authors of textbooks — and teachers as well — must confront the question of whether working toward a precise definition of this concept is justified. In a very large number of cases, such a definition may be given without difficulty (here, we will not discuss the question of how this should be done, but merely point out that, almost always, the precise definition of a concept must grow out of working with the concept rather than precede it). Nonetheless, it should be borne in mind that even the great mathematicians of the past sometimes worked without having at their disposal definitions that we would consider precise according to today's standards (for example, of a limit).

Attempts to sustain high standards of deductive logic, approximating the standards of modern science, can hardly be considered successful. Schools have rejected them — theorems that were too difficult were simply not proven in practice, and as a result the level of deductive logic fell rather than rose. The school course in geometry is not a course in the foundations of geometry. The highest level of deductive logic that is feasible in the classroom is the one that should be aimed at, and this should be done by giving teachers and students difficult problems — difficult but not impossible. The balance of mathematical and pedagogical considerations will be different in each situation and depend on numerous social circumstances.

6.2 *Visual and Informal Geometry in the Study of Three-dimensional Geometry in Basic Schools*

"Draw different polyhedra with five vertices." In order to solve this problem, students do not need, as we have already pointed out above, a formal definition of a polyhedron or a long discussion on what is meant by the word "different" (this can always be explained if necessary). Meanwhile, this problem is useful for developing students' spatial notions and their mathematical imagination in general.

Such a problem can be given to a seventh grader and sometimes even to a sixth grader. Today's programs assign a place to such problems, such as in grades 5 and 6, when covering the topic "Visual Geometry." The textbook by Dorofeev and Sharygin (2002), for example, acquaints students with the concept of axial symmetry and symmetry with respect to a plane, asks them to think about why a right parallelepiped always has three planes of symmetry, and even asks them to investigate whether the plane that passes through the diagonals of the opposite faces of a cube is the cube's plane of symmetry. No formal proofs are given here, and a great deal simply relies on pictures, but even so some deductive arguments emerge.

The informal element must play a role in subsequent studies as well. One of the advantages of geometry is that it is a field in which it is natural to give (literally, to show) examples, to think about which pictures are possible and which pictures are impossible, to make models with one's own hands — again, literally — and thus to overcome the abstractness of mathematics, and so on. All of this must be done not only in grades 5 and 6, but also in all subsequent grades.

The informal element, including the informal study of three-dimensional geometry, has had a complicated history in Russian schools. In the first years after the Revolution, following the recommendations of the international reform movement, it received a great deal of attention. Then it was sharply scaled down (almost destroyed), on the grounds that it was not able to give the children any sound knowledge, but only distracted them from the main thrust of the course (Karp, 2010). Today, attention is again returning to the informal study of geometry in general, and to the early and informal study of solid geometry in particular.

How should this be done, however? How should visual representations be developed without forfeiting deductive logic? How should solid geometry be introduced early on in the course without weakening attention to plane geometry? As we have already pointed out, one approach has been simply to add a solid geometry chapter to the course in plane geometry. There have also been attempts to combine the two courses, as described above. A good teacher will never pass up an opportunity to show the figure that is being studied, even a two-dimensional one, in the surrounding world, which is a three-dimensional world — thus automatically connecting the planar with the spatial. In any case, if the study of geometry has been "rigorous" for a thousand years, then attempts to study it informally at the school level have a far shorter history. Meanwhile, informal study is in many respects no less important, both as preparation for formal study and as a way of developing students.

6.3 *The New and the Old in the Teaching of Geometry*

Alexandrov was, as we have already noted, a committed supporter of the classic geometric method, which goes back to Euclid. Nonetheless, he formulated proofs that were fundamentally new in school geometry. One of them is given below.

The classic school theorem "a line L that is not perpendicular or parallel to plane P (an inclined straight line) is perpendicular to a line M in plane P if and only if the projection of L onto plane P is perpendicular to M" has usually been proven using congruent triangles. In Kiselev's textbook, for example, this was done as follows (Fig. 4):

> Let $\overline{AB}$ be a perpendicular to plane P, $\overline{AC}$ an inclined straight line, and $\overline{BC}$ the projection of that straight line onto plane P. On the straight line, let us mark off equal segments $\overline{CE}$ and $\overline{CD}$ from point C and connect points D and E with points A and B. Now we can see

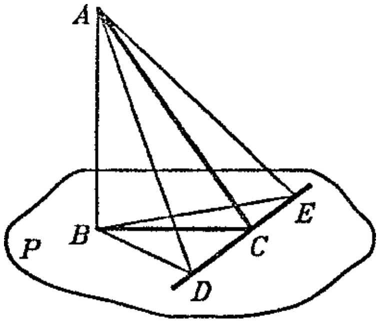

Fig. 4.

> that if $\overline{AC} \perp \overline{DE}$, it follows that ADE is an isosceles triangle, from which in turn follows the congruence $BD = BE$, and because of the properties of an isosceles triangle this means that $\overline{BC} \perp \overline{DE}$. If it is given that $\overline{BC} \perp \overline{DE}$, practically the same argument leads to the conclusion that $\overline{AC} \perp \overline{DE}$.

In place of this proof, Alexandrov suggested the following argument, which is based on the notion of distance and the following proposition: the minimum value of the distance from point A, lying outside a straight line, to the points of this straight line is found at a point that is the base of the perpendicular dropped from A to this straight line.

> Let us take a variable point X on the given straight line and consider the two values AX^2 and BX^2. The triangle ABX is a right triangle. Therefore, $AX^2 = AB^2 + BX^2$. Therefore, the values AX^2 and BX^2 differ by a constant term. Therefore, these quantities have their least values simultaneously — for the same point, X. If X is the base of a perpendicular dropped from A, then it is also the base of the perpendicular dropped from B and vice versa. (Fig. 5)

What is important is not so much that Alexandrov's proof is shorter than Kiselev's (for students, the former is unlikely to be easier than the latter), but that it makes it possible to understand in a new way the essence of a classic theorem — that the theorem is about shortest distances — and in this capacity may be applied and generalized.

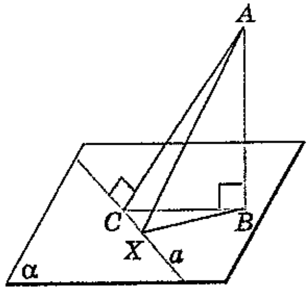

Fig. 5.

In giving this new proof, Alexandrov proceeded as a modern geometer, who is not confined to thinking in terms of Euclid's categories and methods. Attempting to generalize what has happened over the course history, one could say, with all the necessary qualifications, that in school-level instruction there has been a strand oriented toward the *geometry of figures* and another strand oriented toward the *geometry of functions*. The former — which stems, for example, from Euclid — finds the basic content of the subject to consist of the examination and study of the various figures that surround us and their interrelations; the latter, which stems from Klein, and in a certain sense from Descartes, pays the greatest attention to the functions that are important in geometry — geometric transformations. It is likely not by accident that Kolmogorov, who contributed possibly to all branches of 20th century mathematics, inclined toward the latter approach, which connects geometry with other mathematical disciplines, while the geometers Alexandrov and Pogorelov probably found greater affinity with the former, purely geometric approach.

In saying this, however, we must stress that talking about the purity of an approach, so to speak, is completely out of place in this context. The attempt to transform school geometry into a part of some general mathematical theory about functions, matrices, and so on — although it might gladden the research mathematician due to its generality — deprives the student of the experience of direct investigation and reasoning. On the other hand, it would be strange to deliberately conceal from the students the new understanding that has come from the development of science.

What is old, traditional, and Euclidean is supplemented in Russian textbooks with what is new and post-Euclidean. This is accomplished in various ways, and one can argue about the relationship and balance between these two sides of the curriculum. Transformations, vectors, and coordinates, in the opinion of the authors of this chapter, must have a definite place in the school course, although second-generation standards devote little attention to them. On the other hand, we also believe that studies should begin, as history did, not with these materials, but with Euclidean methods. But what is perhaps more important than adding comparatively or even genuinely new sections to the traditional material is to read the classic material in a new way.

The degree to which it will be possible to connect traditions accumulated over the centuries with new mathematical conceptions and new pedagogical and social demands will define the development of school geometry in Russia in the 21st century.

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4
On Algebra Education in Russian Schools

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1 Algebra as a School Subject

Algebra as a science has undergone a whole series of transformations, which have radically changed its content. For Newton, algebra was "the universal arithmetic," which used letter notations to solve arithmetic problems. For Bertrand, "algebra is aimed at shortening, making more precise, and in particular simplifying the solutions of questions that can be posed concerning numbers" (Goncharov, 1958, p. 41). For Lagrange, "algebra may be seen as the science of functions; however, in algebra only those functions are investigated which derive from arithmetic operations, generalized and transposed into letters" (Goncharov, 1958, p. 41).

By the end of the 19th century, the view of algebra as the study of integral rational functions became firmly established in science; it was from this perspective that the university course in "Advanced Algebra" was taught. By the middle of the 20th century, the science of algebra had taken a new step: relying on set-theoretical premises and using the axiomatic approach, it declared its problem to be "the study of 'arithmetic' operations, performed on objects of an arbitrary nature ('group,' 'ring,' 'field')" (Goncharov, 1958, p. 41).

The content of "Algebra" as a school course studied in Russian schools includes the foundations of the science of algebra in each of the interpretations given to it by Newton, Lagrange, and Bertrand; however, at the present time the meaning of the subject "Algebra" is not limited to these interpretations. Of course, the school course in algebra does not involve studying algebra at the level of operations on abstract objects. But its contents are expanded due to the inclusion of the foundations of related branches of mathematics. At present, "Algebra" as an academic subject studied in grades 5–9, as well as its sequel "Algebra and Elementary Calculus," which is studied in grades 10–11, constitute "conglomerate" subjects, addressing basic probability theory, calculus, and analytic geometry, no less than basic algebra.

These school subjects acquired such a form mainly as a result of the influence of the content of education in institutions of higher learning, which is increasingly becoming a form of mass education among young people, and in which calculus, analytic geometry, and probability theory invariably occupy places of paramount importance.

2 The Algebraic Component in the System of School Mathematics Education

Mandatory universal education in Russia consists of three stages: elementary schools (grades 1–4) for children between the ages of six and 10; basic schools (grades 5–9) for children between the ages of 11 and 15; and high or senior schools (grades 10–11) for students who are 16–17 years old. The last stage of education introduces a furcation, which means in particular that students have the opportunity to study subjects of the mandatory sequence — of which, naturally, mathematics is a part — at two different levels: basic and advanced ("profile").

Broadly speaking, the algebraic component is represented at all stages of education. In mathematics classes in elementary school, students gradually learn to use letters to denominate numbers, to put together elementary equations, and so on. In this way, when they reach basic school, they already have a certain minimal experience of "interacting" with letters.

School subjects at the basic and high school stages of education — "Algebra" and "Algebra and Elementary Calculus" — contain, as has already been noted, several different strands of content. A special place among them is occupied by educational material that can in essence be regarded as strictly algebraic. Possessing a history of almost three centuries, it is today isolated into an independent component that interacts with the other components of the course, which have their own aims and goals. This is algebra in its classic definition — computations and equations involving letters.

It is precisely this purely algebraic component of the school course "Algebra," which aims to develop the ability to design mathematical models, abstract from inessential details, and form skills pertaining to the formal manipulation of numerical and literal data, in keeping with the essence of mathematical science that produces that mathematical apparatus without which it is impossible either to investigate problems internal to mathematics or to solve practical and applied problems. It is precisely this purely algebraic component, therefore, that is called upon to demonstrate to students the power of the mathematical method.

3 The Content of Algebra Education in Russian Schools

The content of mathematics education in contemporary Russian schools is prescribed by two main documents approved by the Ministry of Education and Science. These are the basic time allocation plan and the federal component of the educational standards for general education — *Standard* (Ministry, 2004a, 2004b).

The basic time allocation plan allocates no fewer than 875 hours for the study of mathematics in basic school, estimating five hours per week in grades 5–9, and in high school four class hours at the basic level and six class hours at the advanced level. Thus, there are 280 and 420 class hours in all, respectively (the time for the study of the advanced course may be increased up to 12 hours per week by using the school's allotment of elective courses). Of these, about 350 hours are designated for the study of algebraic material in basic school, and in high school about 90 hours at the basic level and no fewer than 140 hours at the advanced level.

The Standard defines the objectives of studying each school subject, a mandatory content minimum, and requirements for graduation.

In basic school, the study of algebraic material is aimed at the formation of a mathematical apparatus for solving problems drawn from mathematics, related subjects, surrounding reality, the acquisition of practical skills necessary for everyday life, and the creation of a foundation for the further study of mathematics. It is intended to facilitate logical development and the formation of the ability to use algorithms. The language of algebra underscores the significance of mathematics as a language for constructing mathematical models of real-world processes and phenomena. One of the basic purposes of studying algebra is to develop students' algorithmic thinking, which is indispensable, for example, for the assimilation of the course in computer science; the study of algebra also helps students acquire the skill of deductive reasoning. The manipulation of symbolic forms contributes in its own specific way to the development of imagination and the capacity for mathematical creativity.

In the process of assimilating algebraic material, students have the opportunity to acquire the symbolic language of algebra, to develop formal-operational algebraic skills, and to learn to apply them in solving mathematical and nonmathematical problems. In a broader context, algebra, along with the other components of school-level mathematics education, facilitates the development of logical thinking, speaking skills, and an understanding of the concepts and methods being studied as crucial means for the mathematical modeling of real-world processes and phenomena (Ministry, 2004c, p. 2).

In the study of algebra *in high school at the basic level*, educators solve the problem of teaching students new types of formulas; improving practical skills and computational literacy; expanding and improving the algebraic apparatus formed in basic school; and using it to solve mathematical and nonmathematical problems (Ministry, 2004d, p. 2).

In high school at the advanced level, the content of algebraic education represented in basic school is developed in the direction of constructing a new mathematical apparatus based on an expansion of number sets from real to complex numbers; and the development and improvement of techniques for algebraic transformation, solving equations, inequalities, and systems.

The study of algebraic material, along with the other components of school-level mathematics education, contributes to solving the general problem of developing students' ability to construct and investigate elementary mathematical models in solving applied problems, and solving problems from related disciplines, thereby increasing knowledge about the special characteristics of the application of mathematical methods in the study of processes and phenomena in nature and society (Ministry, 2004e, pp. 1–2).

As already stated, the content of school-level mathematics education is defined by the Standard. Following the Standard, we offer a description of the content of the algebraic component at the basic school and high school stages; material that must be studied but is not part of the graduation requirements is indicated in italics (Ministry, 2004a, 2004b).

Grades 5–9

Algebraic expressions. Literal expressions (expressions with variables). Numerical values of literal expressions. Permissible values of variables in algebraic expressions. Substituting expressions in place of variables. The equality of literal expressions. Identities; proving identities. Transformations of expressions.

The properties of powers with integer exponents. Polynomials. Adding, subtracting, multiplying polynomials. Short multiplication formulas: squares of sums and squares of differences; *cubes of sums and cubes of differences.* Factoring polynomials. The quadratic trinomial. *Completing the square of a quadratic trinomial.* Viète's theorem. Linear factorization of the quadratic trinomial. Polynomials with one variable. Powers of polynomials. Roots of polynomials.

Algebraic fractions. Reducing fractions. Operating with algebraic fractions.

Rational expressions and their transformations. Properties of square roots and their use in computation.

Equations and inequalities. Equations with one variable. Roots of equations. Linear equations. Quadratic equations: the quadratic formula. Solving rational equations. Examples of solutions to higher-degree equations; methods of variable substitution, factorization.

Equations with two variables; solving equations with two variables. Systems of equations; solving systems of equations. Systems of two linear equations with two variables; solving by substitution and algebraic addition. Equations with several variables. Examples of solutions to nonlinear systems. Examples of solutions to equations in integers.

Inequalities with one variable. Solving such inequalities. Linear inequalities with one variable and systems of such inequalities. Quadratic inequalities. *Examples of solutions to fractional–linear inequalities.*

Numerical inequalities and their properties. *Proving numerical and algebraic inequalities.*

Transposing verbal formulations of relations between magnitudes into algebraic formulations. Solving word problems algebraically.

Representing numbers using points on the number line. The geometric meaning of the absolute value of a number. Numeric intervals: interval, segment, ray.

Cartesian coordinates in the plane; the coordinates of a point. The equation of a straight line, the slope of a line, conditions for parallelism. The equation of a circle centered at the origin and *at any given point.*

Graphically interpreting equations with two variables and systems of such equations; inequalities with two variables and systems of such inequalities.

Grades 10–11

Basic Level

Algebraic expressions. Roots of power $n > 1$ and their properties. Powers with rational exponents and their properties. *The concept of a power with a real exponent.* The properties of powers with real exponents.

The logarithm of a number. *The fundamental logarithmic identity.* Logarithms of products, quotients, powers; *conversion to a new base.* Common and natural logarithms, the number e.

Transformations of elementary expressions containing arithmetic operations, powers, and logarithms.

Equations and inequalities. Solving rational, exponential, logarithmic equations and inequalities. Solving irrational and trigonometric equations.

Basic techniques for solving systems of equations: substitution, algebraic addition, introducing new variables. The equivalence of equations, inequalities, systems. Solving elementary systems of equations with two unknowns. Solving systems of inequalities with one variable.

Using the properties and graphs of functions in solving equations and inequalities. The interval method. Representing the solution sets of equations and inequalities with two variables, and systems of such equations and inequalities, in the coordinate plane.

Applying mathematical methods to solve substantive problems from other areas of science and life. Interpreting results, taking real limitations into account.

Grades 10–11

Advanced Level

Algebraic expressions. The divisibility of integers. Division with a remainder. *Congruences.* Solving problems with integer unknowns.

Complex numbers. The geometric interpretation of complex numbers. The real and imaginary parts, the absolute value, and the argument of a complex number. Algebraic and trigonometric notation for complex numbers. Arithmetic operations on complex numbers in different forms of notation. Conjugate complex numbers. *Raising to a natural power (de Moivre's formula). The fundamental theorem of algebra.*

Polynomials with one variable. The divisibility of polynomials. Dividing polynomials with a remainder. The rational roots of polynomials with integer coefficients. Solving integral algebraic equations. *The Horner scheme.* Bézout's theorem. The number of roots in a polynomial. Polynomials with two variables. Short multiplication formulas for higher powers. Newton's binomial theorem. *Polynomials with several variables, symmetric polynomials.*

Roots of power $n > 1$ and their properties. Powers with rational exponents and their properties. The concept of a power with a real exponent. The properties of powers with a real exponent.

The logarithm of a number. *The fundamental logarithmic identity.* Logarithms of products, quotients, powers; *conversion to a new base.* Common and natural logarithms, the number e.

Transformations of elementary expressions containing arithmetic operations, powers, and logarithms.

Equations and inequalities. Solving rational, exponential, logarithmic equations and inequalities. Solving irrational and trigonometric equations *and inequalities.*

Basic techniques for solving systems of equations: substitution, algebraic addition, introducing new variables. The equivalence of equations, inequalities, systems. Solving systems of elementary equations with two unknowns. Solving systems of inequalities with one variable.

Proving inequalities. The inequality of the arithmetic and geometric means of two numbers.

Using the properties and graphs of functions in solving equations and inequalities. The interval method. Representing the solution sets of equations and inequalities with two variables, and systems of such equations and inequalities, in the coordinate plane.

Applying mathematical methods to solve substantive problems from other areas of science and life. Interpreting results, taking real limitations into account.

After studying algebra in basic school, students must be able to:

- form literal expressions and formulas based on the conditions given in problems; perform number substitutions in expressions and formulas, and carry out the corresponding computations; substitute one expression for another; express one variable in formulas in terms of the others;
- perform basic operations with powers with integer exponents, with polynomials, and with algebraic fractions; factor polynomials; perform identity transformations of rational expressions;
- solve linear, quadratic equations, and rational equations that can be reduced to them, systems of two linear equations and simple nonlinear systems;
- solve linear and quadratic inequalities with one variable and systems of such inequalities;
- represent numbers as points on the number line;
- determine the coordinates of a point in the plane, to construct points with given coordinates; represent the set of solutions to a linear inequality;

use the acquired knowledge and skills in order to:

- use formulas to perform computation tasks, and derive formulas that express dependencies between real magnitudes;
- model practical situations and study the constructed models using the algebraic apparatus;
- describe dependencies between physical magnitudes through appropriate formulas when investigating simple practical situations.

After studying algebra in high school at the basic level, students must be able to:

- use known formulas and rules in order to transform literal expressions that contain powers, radicals, and logarithms;
- compute the values of algebraic expressions, performing the necessary substitutions and transformations;
- solve equations and elementary systems of equations, using *the properties of functions* and their graphs;
- solve rational, exponential, and logarithmic equations and inequalities, elementary irrational and trigonometric equations, and systems of such equations;
- form equations and *inequalities* based on the conditions given in a problem;
- use the graphical method to obtain approximate solutions to equations and inequalities;
- represent the solution sets of elementary equations and systems of such equations in the coordinate plane;

use the acquired knowledge and skills in order to:

- perform practical computation tasks using formulas, including formulas that contain powers, radicals, logarithms, and trigonometric functions, relying on reference materials and simple computing devices if necessary;
- construct and investigate elementary mathematical models.

After studying algebra in high school at the advanced level, students must be able to:

- use concepts connected with the divisibility of integers in order to solve mathematical problems;

- find the roots of polynomials with one variable, to factor polynomials;
- carry out transformations of numerical and literal expressions that contain powers, radicals, logarithms, and trigonometric functions;
- solve equations, systems of equations, and inequalities, using the properties of functions and their graphic representations;
- prove simple inequalities;
- solve word problems by formulating equations and inequalities, and take into account the limitations specified in the conditions given in the problems when interpreting the results;
- represent the solution sets of equations and inequalities with two variables, and of systems of such equations and inequalities, in the coordinate plane;
- use the graphical method in order to find approximate solutions to equations and systems of equations;
- solve equations, inequalities, and systems of equations and inequalities by relying on graphic representations, the properties of functions, and derivatives;

use the acquired knowledge and skills in practical activities and everyday life in order to:

- perform practical computation tasks using formulas, including formulas that contain powers, radicals, logarithms, and trigonometric functions, relying on reference materials and simple computing devices if necessary;
- use functions to describe and investigate real-world dependencies, representing them graphically; interpret the graphs of real-world processes;
- construct and investigate elementary mathematical models.

4 Methodological Issues in Teaching Algebra

4.1 *Basic School (grades 5–9; students aged 10–15)*

4.1.1 *An Overview*

The algebraic material presented above is arranged in two stages, which correspond to the distinctive features of the cognitive activity of students of ages 10–11 and 12–15: these are grades 5–6 and grades 7–9. In grades 5–6, algebraic questions are included in an integrated course in mathematics, in which students continue to study positive integers and are introduced to fractions and decimals, positive and negative numbers, computational techniques, and elementary geometric concepts. In grades 7–9, algebraic questions are examined in a course which, in contrast to the course for grades 5–6, is considered a systematic course. It is called "Algebra," although, as has already been stated above, strictly algebraic material forms only a part of its content.

The Standard does not require that the content be distributed into these two stages. The requirements formulated in the Standard pertain to the outcome of the course, i.e. they indicate the objectives that must be met by the end of ninth grade, without prescribing the objectives of the first stage. This makes it possible for schools to use different systems of textbooks, all of which meet the Standard's requirements, but which differ from one another in their methodological approaches and the way they distribute the material between the two stages — and these differences can be substantial.

To convey an idea of the various approaches to presenting algebra in basic school, in the following we will examine, and when necessary compare, approaches that are embodied in two systems of textbooks. One of them consists of textbooks by Vilenkin *et al.* (2007, 2008) and Makarychev *et al.* (2009a, 2009b, 2009c) for grades 5–6 and 7–9, respectively. Although these textbooks were created by different teams of contributors, certain connections exist between them, and they are often used in succession in teaching practice as part of the same sequence. Also crucial is the circumstance that both of their first editions were prepared in the 1970s on the basis of the same pedagogical ideology, which was put forward during a period of radical reforms in mathematics education, whose ideological leader was the academician Andrey Kolmogorov. It must be said that in its time this series of textbooks made a significant progressive contribution to the system of mathematics education in the schools of our country. Here, we will not discuss all of the innovations that they introduced or their numerous positive aspects, since this is not the subject of this chapter. We will merely note that many ideas developed by the authors of these textbooks became the classic heritage of Russian methodology and have preserved their relevance to this day. However, as far as the algebraic material is concerned, in our opinion, from the point of view of contemporary pedagogy, the solution offered in these textbooks is not optimal.

Over the intervening decades, these textbooks have been repeatedly reworked in accordance with changes in programs, which followed certain international trends. But these changes had the least impact on algebra, and the approaches to its presentation have not undergone any radical change. At present, these textbooks are in high demand in schools. Of all textbooks, they are the most widely used, although this may be due in part to the conservatism of teachers and their adherence to established traditions.

The other group consists of textbooks by Dorofeev, Sharygin, *et al.* (2007a, 2007b) as well as Dorofeev, Suvorova, *et al.* (2005, 2009a, 2009b), which are also intended for teaching in grades 5–6 and 7–9. They were prepared by the same team of contributors, and therefore the connections between them were planned from the outset. These textbooks were developed in the 1990s, a period of social and ideological changes in the country, which necessarily impacted the aims and paradigms of school education. They reflect a different approach to presenting algebraic material, which is based, on the one hand, on their authors' views concerning mathematics education, and, on the other hand, on the experience gained from using traditional textbooks in education.

It may be said that these two sets of textbooks constitute conspicuous examples of two different ideologies in mathematics education, particularly in teaching algebra.

4.1.2 *Algebra for students of ages 10–12 (grades 5–6)*

The material that usually pertains to grades 5–6 is labeled in curricula as "Elements of Algebra." The objectives that its study is meant to meet are defined differently by different teams of contributors.

The main purpose of studying algebra, as defined by the authors of the textbooks by Vilenkin *et al.* (2007, 2008), is the formation of basic formal-operational skills. Of the basic content described above, this series of textbooks selects the following items for this stage of education: literal expressions, numerical values of literal expressions, using literal notation to indicate the properties of arithmetic operations, transformations of literal expressions (multiplying a sum by a number, factoring out common factors, combining like terms, simplifying products, and removing parentheses that have a plus or minus sign in front of them), solving equations, and using algebraic methods to solve word problems.

In their methodological approaches, the authors rely substantially on the fact that the students have already acquired some experience in working with letters in elementary school. Therefore, they begin using letter symbolism quite extensively in the very first classes of fifth grade without any kind of special discussion about mathematical language and, in particular, about the meaning of literal expressions. The course makes no explicit distinction between concrete, visual arithmetic and formal, abstract algebra. One might say that in some sense the students study "algebraized arithmetic." Each time new types of numbers are introduced and new computational algorithms are examined, the students also carry out assignments that involve manipulating literal expressions and solving equations, in which this new knowledge is used. Below, we offer examples of this concurrent development of the arithmetic and algebraic components of the course, reflected in exercises from the textbook by Vilenkin *et al.* (2008).

Arithmetic	Algebra
I. Ordinary fractions	
Compute using the distributive property: $8\frac{5}{11} \cdot 4\frac{2}{9} + 8\frac{5}{11} \cdot 6\frac{7}{9}$. (p. 88)	Simplify the expression $\frac{5}{18}x + \left(\frac{5}{12}x - \frac{1}{4}x\right)$. Simplify the expression $\frac{13}{15}m - \frac{3}{4}m + \frac{1}{12}m$ and find its value when $m = 2\frac{1}{2}; 6\frac{1}{4}$. Solve the equation $\frac{7}{12}m + \frac{2}{3}m - \frac{1}{4}m = 7$. (p. 89)

Arithmetic	Algebra
Divide: $\frac{3}{5} \div \frac{9}{25}$; $3\frac{7}{39} \div 1\frac{5}{31}$. (p. 98)	Represent the following quotient in the form of a fraction: $\frac{m}{n} \div \frac{a}{k}$; $b \div \frac{c}{n}$. (p. 98) Solve the equation (a) $y \div 1\frac{1}{2} = 2\frac{1}{3} \cdot \frac{1}{3}$; (b) $3\frac{1}{2}\left(\frac{2}{3}x + \frac{4}{7}\right) = 2\frac{1}{3}$. (p. 100)
	II. Decimal fractions
Find the value of the expression $102{,}816 \div (3.2 \cdot 6.3) + 3.84$. (p. 67).	Simplify the expression $3.7x + 2.5y + 1.6x + 4.8y$. (p. 76) Solve the equation $9.5x - (3.2x + 18x) + 3.75 = 6.9$. (p. 78)
Compute $\frac{0.2 \cdot 6.2 \div 0.31 - \frac{5}{6} \cdot 0.3}{2 + 1\frac{4}{11} \cdot 0.22 \div 0.01}$. (p. 112)	Find the value of the expression (a) $\frac{2x}{y} - \frac{x}{2y}$ when $x = 18.1 - 10.7$, $y = 35 - 23.8$; (b) $\frac{a}{5.7 - 4.5} + \frac{a}{2.8 + 4.4}$ when $a = 2\frac{1}{7} + 1\frac{4}{5}$. (p. 112)
	III. Ratios and proportions
Find the ratio of 0.25 to 0.55. (p. 118)	The length of a rectangle is a cm and its width is b cm. The length of another rectangle is m cm and its width is n cm. Find the ratio of the area of the first rectangle to the area of the second rectangle. Find the value of the obtained expression if $a = 6.4$, $b = 0.2$, $m = 3.2$, $n = 0.5$. (p. 123)
	IV. Positive and negative numbers
Perform the following operations: $-6 \cdot 4 - 64 \div (-3.3 + 1.7)$. (p. 198)	Find the value of the expression $(3m + 6m) \div 9$, if $m = -5.96$. Solve the equation $-\frac{4}{7}y = \frac{8}{21}$. (p. 198)

Arrange the terms in a convenient order and find the value of the expression $-6.37 + 2.4 - 3.2 + 6.37 - 2.4$. (p. 208)	Simplify the expression $6.1 - k + 2.8 + p - 8.8 + k - p$. (p. 208)
Remove the parentheses and find the value of the expression $-6.9 - (4.21 - 10.9)$. (p. 216)	Simplify the expression $-a - (m - a + p)$; $m - (a + m) - (k + a)$. (p. 217) Write down the difference of the two expressions $-p - a$ and $k - a$, and simplify it. (p. 217) Solve the equation $7.2 - (6.2 - x) = 2.2$. (p. 217)

The solutions to the equations reproduced in this table are based on arithmetic techniques: students solve them by relying on facts about dependencies between the components of operations, which are expressed in rules for finding unknown terms, minuends, divisors, and so on. At the same time, these types of equations have a fairly high level of difficulty.

Note that the subject of equations involves not only using algorithms, but also using the algebraic method to solve word problems. Students solve a considerable number of word problems by forming equations. The problems' algebraic component is developed in parallel with the formation of students' operational abilities and is connected with the content of arithmetic problems. We will illustrate this by providing examples of problems solved by sixth graders:

(1) In order to make sour cherry jam, one must combine two parts cherries with three parts sugar (in mass). How many kilograms of sugar and how many kilograms of cherries must be used in order to obtain 10 kg of jam if its mass is reduced by 1.5 times during cooking?

[Equation: $3x + 2x = 10 \cdot 1.5$, where x is the mass of one part in kilograms.]

(2) Three boxes contained 76 kg of sour cherries. The second box had twice as many sour cherries as the first, while the third contained 8 kg more sour cherries than the first. How many kilograms of sour cherries were in each box?

$$\text{[Equation: } x + 2x + (x + 8) = 76, \text{ where } x \text{ is the mass of sour cherries in the first box, in kilograms.]}$$

(3) The arithmetic mean of four numbers is 2.75. Find these numbers if the second is 1.5 times greater than the first, the third is 1.2 times greater than the first, and the fourth is 1.8 times greater than the first.

$$\text{[Equation: } (x + 1.5x + 1.2x + 1.8x) \div 4 = 2.75, \text{ where } x \text{ is the first number.]}$$

(4) A father is $3\frac{1}{3}$ times older than his son, while the son is 28 years younger than his father. How old is the father and how old is the son?

$$\left[\text{Equation: } 3\frac{1}{3}x - x = 28, \text{ where } x \text{ is the son's age.}\right]$$

This organic integration of arithmetic and algebra concludes with a certain systematization of the algebraic material: an examination of strictly algebraic questions — removing parentheses, the coefficient, like terms, and solving equations. The solving of equations is now grounded in the use of rules for equivalent transformations of equations (the word "equivalence" — which in Russian textbooks is reserved for logical equivalence only — is, of course, not used at this stage). Here, the students deal with formal algebra, and the level of the transformations presented to them is quite high.

In this way, these textbooks achieve rather close integration of arithmetic and algebraic material. However, teaching experience points to a number of negative consequences arising from such early and insistent "algebraization." First, this approach to some extent hinders the formation and development of practically oriented arithmetic skills, such as the use of percentages in real-life situations. While students formally assimilate the central topics of arithmetic — fractions and decimals — their computational skills suffer. A considerable percentage of students is unable to compare fractions or put them in ascending order, to shift from one form of fractional notation to another. This is revealed by both national and international studies. Thus, many students have difficulty with the following types of problems:

- Which of the following numbers is the smallest: $\frac{1}{6}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{2}$.
- Which of the following numbers is contained between the numbers 0.07 and 0.08? 0.0075, 0.6, 0.075, 0.75.
- Find the ratio of the numbers 0.5 and 0.3.

Setting the formation of formal-operational skills pertaining to the transformation of literal expressions as a central objective, the authors rise to a sufficiently high level of such transformations, exceeding the capacities of a considerable number of 12-year-old children. Schoolchildren are not always able to handle much easier problems than those which they solve in class (see the table above). For example:

- Solve the equation $\frac{1}{2}x = 6$.
- Which of the following expressions is equal to the sum $a + a + a + a$?
  (1) $a + 4$, (2) a^4, (3) $4a$, (4) $4(a + 1)$.

As a consequence, the textbooks of the following stage (Makarychev *et al.*, 2009a, 2009b, 2009c) do not begin at the level set by the textbooks of Vilenkin *et al.* In terms of the transformations of algebraic equations that the students are asked to carry out and the equations that they are asked to solve, the first classes in algebra at the following stage of education (grade 7) do not constitute a natural continuation of what has come before; in these classes, everything begins anew.

The key feature of the second set of textbooks for this stage of schooling (Dorofeev, Sharygin *et al.*, 2007a, 2007b) stems from the emphasis that they place on the arithmetic and algebraic components of the course: the balance in them has shifted in favor of the former. A greater role is now played by arithmetic, the study of number systems, computational algorithms; most importantly, the course relies extensively on using arithmetic methods to solve word problems, which is seen as an effective way to facilitate the students' logical development.

At the same time, the approach to presenting algebraic material is fundamentally altered as well. The quantity of formal "algebraic" work is substantially reduced; the very purpose of studying this material is different. One might say that at the center of attention is the role of letters as elements of mathematical language. First of all, the letter acts as the "name" of any number in some set. This is underscored in formulations that use quantifying phrases such as "for any..." and "for all...." Consider the following example of a text that is read by students in fifth grade:

> You know the commutative property of addition: when the places of terms are switched, the sum does not change. In accordance with this property, for example,
>
> $$280 + 361 = 361 + 280, \quad 0 + 127 = 127 + 0.$$
>
> Using letters, the commutative property can be written in the following way:
>
> | For any numbers a and b, $a + b = b + a$. |
> |---|
>
> This *literal equality*, which expresses a general property of the addition of numbers, has replaced for us an infinite number of number equalities (Dorofeev, Sharygin *et al.*, 2007a, p. 82).

Similar arguments are presented in introducing literal notation for the commutative property of multiplication, the associative property of addition and multiplication, and so on.

The letter may also act as a proper noun. For example, π is a quite definite number, about which the students so far know only that it is a number of a new kind, which is neither an integer nor a fraction, and that it may be expressed approximately in decimals. Special letters are "assigned" to the sets of natural numbers, integers, and rational numbers — N, Z, and Q, respectively.

Students learn the rules for writing literal expressions, in particular the role of parentheses as a "grouping" sign. Classroom activity is mainly aimed at getting the students to learn and grasp the significance of and reasons for introducing letters, and to practice "translating" from Russian into mathematical language. Several examples:

1. Write in the form of a mathematical sentence:

   (a) the number k is less than 5; (b) the absolute value of the number m is greater than 1; (c) the square of the number a is equal to 4. (Dorofeev, Sharygin *et al.*, 2007b, p. 244)

2. The following examples illustrate a certain rule. Formulate this rule and write it down using letters:

   (a) $7 \cdot 0 = 0$, $15.3 \cdot 0 = 0$, $\frac{2}{5} \cdot 0 = 0$;
   (b) $4 + (-4) = 0$; $0.3 + (-0.3) = 0$; $\frac{1}{3} + \left(-\frac{1}{3}\right) = 0$. (Dorofeev, Sharygin *et al.*, 2007b, p. 245)

3. In order to write "long" expressions, mathematicians often use an ellipsis. For example, the expression $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 50$ means the product of all natural numbers from 1 to 50. Write down the following in the form of a mathematical expression:

   (a) the product of all natural numbers from 1 to 100;
   (b) the product of all natural numbers from 1 to n;
   (c) the sum of all natural numbers from 1 to 100;
   (d) the sum of all natural numbers from 1 to n. (Dorofeev, Sharygin *et al.*, 2007b, p. 245)

4. Write down the following problem in the form of an equation and solve it:
   Tanya thought of a number, multiplied it by 15, and subtracted the result from 80. She obtained 20. What number did Tanya originally think of? (Dorofeev, Sharygin *et al.*, 2007b, p. 259)

Algebraic "technique" — the transformation of literal expressions — belongs to the next educational stage and begins to be studied systematically in grade 7. But at the stage of grades 5–6, the study of number systems and computational algorithms is organized in such a way as to create a substantive foundation for the study of algebraic transformations in the future: students learn the properties of arithmetic operations as an apparatus for the transformations of numeric expressions. Thus, in the fifth-grade course, students examine the possibility of using the rules of addition and multiplication in order to substitute numeric expressions with simpler expressions whose value may even be found mentally. The problems presented to the children are simple and understandable; the work they do is substantive, motivated, and easy to appreciate. At the same time, the students perform quite serious manipulations with numeric expressions: they write down numeric sequences, group terms and factors in a convenient manner, factor out common factors in numeric sums and products, and so on. Two examples:

Example 1. Students are asked to find the value of the product $4 \cdot 7 \cdot 11 \cdot 25$ (Dorofeev, Sharygin *et al.*, 2007a, p. 84).

They reason in the following manner: the product of 4 and 25 equals 100, and multiplying by 100 is easy, and therefore let us group the factors in the following way:

$$4 \cdot 7 \cdot 11 \cdot 25 = (4 \cdot 25) \cdot (7 \cdot 11) = 100 \cdot 77 = 7700.$$

Example 2. Students are asked to find the value of the fraction $\frac{\frac{1}{3}-\frac{1}{5}}{\frac{2}{3}-\frac{1}{2}}$ (Dorofeev, Sharygin *et al.*, 2007b, p. 11).

To find the value of this expression, the students can perform three operations: find the value of the fraction's numerator, find the value of the fraction's denominator, and divide the former by the latter. But they can also employ a different approach: using the "basic property of fractions" (the fact that multiplying the numerator and the denominator of a fraction by the same number produces a fraction that is equal to the original fraction), they can manipulate the given "multistory" fraction and obtain the answer much more easily and quickly. The students' reasoning is approximately as follows: let us multiply the numerator and the denominator of the fraction by a "convenient" number to get rid of the fractions in the numerator and the denominator. In the given case, this number can be, for example, 30:

$$\frac{\frac{1}{3}-\frac{1}{5}}{\frac{2}{3}-\frac{1}{2}} = \frac{30 \cdot \left(\frac{1}{3}-\frac{1}{5}\right)}{30 \cdot \left(\frac{2}{3}-\frac{1}{2}\right)} = \frac{10-6}{20-15} = \frac{4}{5}.$$

Of course, this solution is presented as an alternative to the first. Although it is demonstrated to all students, the teacher emphasizes that it makes sense to proceed in this way if the intermediate computations can be performed mentally.

Performing transformations of this kind constitutes a good, substantive form of practice, which prepares the students for learning to carry out transformations of literal expressions, which, as has already been noted, are a topic of study at the subsequent stage (grades 7–9) — as is solving equations by using transformations. At this stage, however, the aim of this activity is not so much the development of a skill as the simple process of carrying out such transformations.

4.1.3 *Algebra for students of ages 12–15 (grades 7–9)*

In all textbooks for this stage, including those examined in this chapter, the quantity of algebraic material is practically identical. It is determined by the contents of the corresponding section of the Standard, cited above.

Literal numeration. The presentation of algebraic material at this stage most often begins with a section that can be labeled "Introduction to Algebra." Its content depends substantially on which textbook was used at the previous stage and how much algebraic preparation students received during that period. If the textbooks belonged to the series by Makarychev *et al.*, then they begin with systematization of the knowledge acquired during the preceding stage — students again go over the basic skills connected with combining like terms, removing parentheses, and simplifying products; they are also introduced to such concepts as identity and identity transformations of expressions. Here, too, students again review material connected with solving equations, are introduced to the concept of equivalent equations, and investigate how many solutions an equation of the type $ax + b = 0$ has, depending on the values of the coefficients a and b.

The textbooks by Dorofeev *et al.* begin by listing the properties of arithmetic operations (in literal notation), which are already known to the students, after which the students use numerical examples to write down literal equalities that express certain computational techniques, such as the technique of subtracting a sum from a number: $a - (b + c) = a - b - c$. On this basis, the textbook introduces the concept of equal literal expressions and the concept of the transformation of an expression, which is treated as a replacement of one expression by another that is equal to it. Further, students learn about certain basic transformations of expressions, such as combining like terms, removing parentheses, and simplifying products. In contrast to the previous series of textbooks, here students learn these concepts for the first time; therefore, the presentation includes deductive arguments, which lead to the formulation of a system of rules.

Subsequently, all of the textbooks proceed to a systematically structured study of rational expressions. Polynomials and operations involving polynomials are examined. Students learn that the sum, difference, and product of polynomials can always be transformed into a polynomial (division of polynomials is a topic that belongs to a later stage). Specific attention is devoted to polynomials with one variable (the concept of the root of a polynomial is introduced, and a number of questions connected with polynomials of the type $ax^2 + bx + c$ are examined). The students study techniques for factoring polynomials, factoring out common factors, and grouping, using the formulas $a^2 \pm 2ab + b^2 = (a \pm b)^2$, $a^2 - b^2 = (a - b)(a + b)$, and $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. They are then introduced to the concept of algebraic fractions and examine operations involving algebraic fractions — addition, subtraction, multiplication, and division.

The body of information acquired by the students makes it possible to introduce the concept of integral expressions and fractional expressions, and to pose a question about the possibility of transforming them, respectively, into polynomials and algebraic fractions. The introduction of integral and fractional expressions is accompanied by a discussion on the domain of a rational expression. Quite subtle aspects of this topic are touched on; in particular, it is elucidated that in the transformation of a fractional expression the domain might change (become larger).

Additionally, in connection with transformations of expressions, students study square roots and transformations of numeric expressions containing radicals.

As they assimilate algorithms for transforming expressions, the students are given problems in which these algorithms may be applied. In particular, the transformations studied are always used to solve equations. For example, when they are acquiring the skill of multiplying a monomial by a polynomial, the students solve the following type of equations: $2(x + 5) - 3(x - 2) = 10$. When studying how to factor polynomials, they examine equations that can be solved by relying on the fact that a product equals zero: $(x + 3)(5x - 4) = 0$, $3(x - 2) + (x^2 - 4) = 0$. Identity transformations are used in the course to simplify computations, solve divisibility problems, prove identities, and so on. Several examples:

- Compute: $\frac{53^2-27^2}{79^2-51^2}$. (Makarychev, 2009a, p. 167)
- Compute using the formula $(a-b)(a+b)=a^2-b^2$:
  (a) $201 \cdot 199$; (b) $1.05 \cdot 0.95$. (Makarychev, 2009a, p. 164)
- Find 10 factors of a number equal to $97^2 - 43^2$. (Dorofeev, Suvorova *et al.*, 2005, p. 218)
- Check the following equalities:

$$\frac{1}{2}-\frac{1}{3}=\frac{1}{2\cdot 3},\ \frac{1}{3}-\frac{1}{4}=\frac{1}{3\cdot 4},\ \frac{1}{4}-\frac{1}{5}=\frac{1}{4\cdot 5},\ \frac{1}{5}-\frac{1}{6}=\frac{1}{5\cdot 6}.$$

  Continue this sequence of equalities. Write down the corresponding literal equality and prove it. (Dorofeev, Suvorova *et al.*, 2009a, p. 20)
- Prove that if the side of a square is increased 10 times, then its area will increase 100 times. How many times will the volume of a cube increase if its edge is increased n times? (Dorofeev, Suvorova *et al.*, 2005, p. 162)

Equations and systems of equations. In parallel with the topic of transformations, the textbooks also develop the related topic of equations, inequalities, and systems. The textbooks contain no direct answers to the following question: What is an equation? The term itself is familiar to students from the course in mathematics for grades 5–6. In seventh grade, the students are given a word problem, which is then used as a basis to formulate an equality that contains an unknown magnitude, indicated by a letter. The students are reminded that such an equality is called an "equation," and that in order to obtain the answer to the problem this equation must be solved. For example, the concept of an equation was introduced in the following way in a class observed by the authors of this chapter, in which the textbook of Dorofeev, Suvorova *et al.* (2005) was used (the lesson also aimed to demonstrate conclusively to the students the advantages of the algebraic method over the arithmetic one).

Initially, students were given the following problem:

> A family has two pairs of twins, born three years apart. In 2002, all of them together turned 50. How old was each twin in 2000?

This problem was solved by the students not independently, but in a group discussion organized by the teacher. On the teacher's suggestion, the students solved this problem arithmetically. Using their combined effort, the children solved the problem, but said that it was difficult.

After this, the teacher suggested solving it algebraically, noting that to do so it would be necessary to translate the conditions given in the problem into the language of mathematics. The students reasoned as follows:

> Let x stand for the age of the younger twins in 2000. Then the older twins were $x + 3$ years old in that year. Two years later, the younger twins were $x + 2$ years old, while the older ones were $x + 5$ years old. Thus, we have an equation: $(x+2)+(x+2)+(x+5)+(x+5) = 50$.

After simplifying, we obtain the equation $4x + 14 = 50$, and hence $x = 9$. Thus, the younger twins were 9 years old in 2000, and the older twins were 12 years old. The students felt that this solution was considerably easier.

In this way, a word problem provides the motivation for subsequent formal activity in studying equations and learning algorithms for solving them. A significant portion of this part of the course involves studying equations with one variable. These are integral equations, which, as a result of transformations, are reduced to linear equations of the type $ax+b = 0$, or to quadratic equations of the type $ax^2+bx+c = 0$.

A considerable portion of this part of the course is devoted to the study of quadratic equations, which is a tradition in Russian schools. The quadratic trinomial and quadratic equations serve as a means of introducing the students to certain mathematical ideas. They contain material that is conceptually rich and convenient for organizing cognitive activity, and at the same time corresponds to the capacities of students at this age. The central topic here is the derivation of the quadratic formula; some of its variations are sometimes also examined — the formula for the roots of an equation with an even second coefficient, and the formula for the roots of a reduced quadratic equation ($a = 1$). Along with learning the algorithm for solving equations by using the quadratic formula, the students carry out elementary investigations; for example, they solve problems of the following type:

- Determine the discriminant of the equation $x^2 + 7x - 1 = 0$ and answer the following questions:
(a) Does the equation have roots? (b) If it does, then how many? (c) Are the roots rational or irrational numbers? (Dorofeev, Suvorova *et al.*, 2009a, p. 118)
- Find a value of c for which the equation $5x^2 - 2x + c = 0$ has roots, and find a value of c for which it does not have roots. (p. 118)
- Given the equation $2x^2 - 7x + 3 = 0$, write a new equation, switching the places of coefficients a and c in the given equation. Solve both equations. How are their roots related? (p. 119)

Students examine techniques for solving specific types of quadratic equations, namely incomplete quadratic equations (equations of the forms $ax^2 + bx = 0$ and $ax^2 + c = 0$). The study of quadratic equations concludes with an examination of formulas that connect the roots of a quadratic equation with its coefficients (note that in contrast to many foreign textbooks, in Russia this topic is traditionally studied — with reason — *after* the formulas for solving quadratic equations have been derived). The students use these formulas to find roots mentally and to check whether the solutions to equations are correct. Note, too, that this material is employed in all textbooks as a training ground for solving problems of the most varied levels of difficulty. Problems in these topics, which are given to students in all textbooks as well as in the classroom, no longer serve to develop their skills, but to develop their thinking, to organize interesting mathematical activities, and to expand the arsenal of techniques that are available to the students. These problems cover a broad range of levels of difficulty for different categories of students. Consider the following examples of problems solved by students in class [some of them are taken from the textbook by Dorofeev, Suvorova *et al.* (2009a, p. 134)]:

- Without solving the equation $x^2 + 7x - 1 = 0$, determine whether it has roots, and if it does, what their signs are.
- Find all integer values of p for which the equation $x^2 + px + 15 = 0$ has integer roots.
- Knowing that the quadratic equation $x^2 + px + q = 0$ has roots x_1 and x_2, formulate a quadratic equation that has roots $3x_1$ and $3x_2$.

- Prove that if the sum of the coefficients of the quadratic equation $ax^2 + bx + c = 0$ is equal to 0, then one of the roots of this equation is the number 1. Mentally find the roots of the equation $100x^2 - 150x + 50 = 0$.

Later, when they are close to graduating from basic school, students may solve third- and fourth-degree equations, such as

$$2x^3 - x^2 - 8x + 4 = 0 \quad \text{and} \quad 2x^4 + 9x^2 + 4 = 0.$$

The main purpose of this material in the general education course is to broaden the students' horizons. The historical context that naturally arises in the study of such equations is present in all textbooks in one form or another. For example, in the textbook of Dorofeev, Suvorova *et al.* (2009b), the presentation is organized as follows:

> After a short survey of what the students already know about techniques for solving linear and quadratic equations, they are informed that they will be able to solve higher-degree equations only in certain specific cases, and that already for fifth-degree equations there is no general formula at all. At the beginning of the 19th century, the Norwegian mathematician Niels Henrik Abel proved that it is impossible to obtain the roots of even such a comparatively simple equation as $x^5 + x - 1 = 0$ by using arithmetic operations and finding roots.
>
> For third- and fourth-degree equations, such formulas do exist. The method for solving third-degree equations was discovered by Italian mathematicians in the 16th century. But the formulas for solving third- and fourth-degree equations are so complicated that they are practically never used. Also, in order to use them, one must employ new numbers, so-called complex numbers, which were invented for this purpose. (Dorofeev, Suvorova *et al.*, 2009b, p. 131)

Subsequently, the students are introduced to two techniques for solving third- and by fourth-degree equations by factoring and by introducing a new variable.

In studying equations, students also solve equations that contain a variable in the denominator of a fraction, such as

$$\frac{2x}{x-3} + \frac{6}{(x-3)(x-4)} = \frac{x}{x-4}.$$

In solving such equations, the students for the first time encounter a technique that does not guarantee identity of a transformation. Therefore, a necessary part of this technique is the verification of the obtained solutions.

Another topic of study in this algebraic part of the course is systems of equations with two variables. Students are introduced to the concept of the equation with two variables and its graph. They then go on to develop skills that are associated mainly with the linear equation $ax + by = c$. Systems containing two equations with two variables are solved. The main stress falls on systems of two linear equations, as well as on systems in which one of the equations is a second-degree equation. The students learn techniques for solving systems of equations such as substitution and addition. In more difficult problems, they also investigate more complex systems, whose solutions require the use of certain additional techniques. The presentation of this whole topic is usually permeated with references to graphic illustrations and interpretations: graphs are used for solving systems (in cases where the students lack an appropriate algebraic apparatus for solving them), for determining the number of solutions, and for conducting investigations. In this way, graphs are used to determine how many solutions a system of two linear equations with two variables has.

Examples of typical problems that are solved in class are:

1. Find the coordinates of the points in which the line $2x + 3y = 4$ intersects the coordinate axes, and graph this line.
2. Using the diagram (Fig. 1), write down the system of equations whose solution consists of the following pair of numbers: (a) $(0, -4)$; (b) $(4, 1)$; (c) $(-4, 0)$; (d) $(1, 4)$.
3. Solve the system of equations $\begin{cases} 4x - 3y = -16 \\ 6x + 5y = 14 \end{cases}$.
4. Determine the coordinates of the points of intersection:
   (a) of the lines $8x + y = 27$ and $5x - y = 25$;
   (b) of the line $y = 2x + 4$ and the parabola $y = x^2 - 3x - 10$.
5. Using graphs, determine how many solutions the following system of equations has: $\begin{cases} x^2 + y^2 = 9 \\ xy = 4 \end{cases}$.
6. Solve the following system of equations: $\begin{cases} 2x + y = 1 \\ x^2 + xy = -6 \end{cases}$.

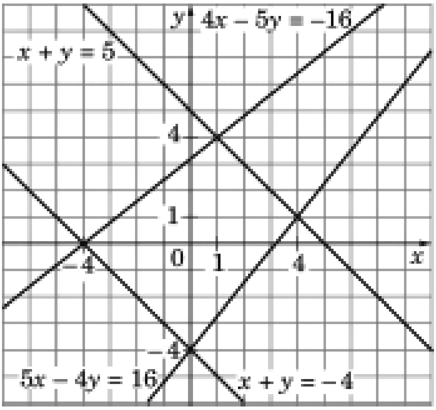

Fig. 1.

7. Solve the following system of equations:

(a) $\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{1}{2}, \\ x + y = 9 \end{cases}$; (b) $\begin{cases} xy - y = 1, \\ xy + x = 4. \end{cases}$

Strong students are introduced to certain special techniques for solving systems. Let us consider how a teacher using the textbook by Dorofeev, Suvorova *et al.* (2009b) can use the same system, $\begin{cases} x^2 + y^2 = 10 \\ xy = -3 \end{cases}$, to demonstrate various techniques for solving it.

Students who use this textbook first solve the given system by employing the method of substitution (in other words, from the second equation they obtain an expression that, for example, denotes the unknown y in terms of the unknown x, and then substitute this expression, $y = -\frac{3}{x}$, in the first equation, which now becomes easy to solve). After several classes, the students may be introduced to other ways of solving this system. The main idea of each of them consists in the fact that the given system can be reduced to several simpler systems.

First method. Let us multiply both sides of the second equation of the system by 2 and add it to the first equation. We obtain the equation $x^2 + y^2 + 2xy = 4$, i.e. $(x + y)^2 = 4$. Together with the second equation, it forms the system $\begin{cases} (x + y)^2 = 4 \\ xy = -3 \end{cases}$.

The equality $(x + y)^2 = 4$ means that $x + y = 2$ or $x + y = -2$.

Thus, the system "breaks down" into two simpler ones:

$$\begin{cases} x + y = 2, \\ xy = -3; \end{cases} \quad \begin{cases} x + y = -2, \\ xy = -3. \end{cases}$$

The solution to the first system consists of the pairs $(-1, 3)$ and $(3, -1)$; the solution to the second consists of the pairs $(1, -3)$ and $(-3, 1)$. Each of these pairs satisfies the original system, and it has no other solutions. Thus, the original system has four solutions: $(-1, 3)$, $(3, -1)$, $(1, -3)$, $(-3, 1)$. In this way, the equations that must be solved are noticeably simpler than the one examined at the beginning.

A graphic illustration helps to better understand the substantive side of the solutions just given. Figure 2 represents a circle and a hyperbola — the graphs of the equations in the system $\begin{cases} x^2 + y^2 = 10 \\ xy = -3 \end{cases}$. They intersect at four points.

Figure 3 illustrates, within the same system of coordinates, the graphic solutions to the systems $\begin{cases} x + y = 2 \\ xy = -3 \end{cases}$ and $\begin{cases} x + y = -2 \\ xy = -3 \end{cases}$.

Each line intersects the hyperbola $xy = -3$ at two points, and we have four points of intersection in all. These are the same points that were formed by the intersections of the hyperbola and the circle.

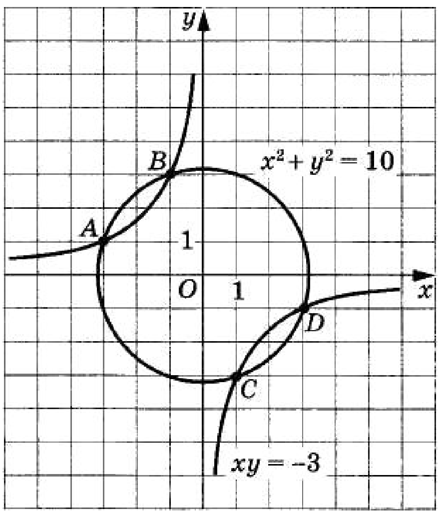

Fig. 2.

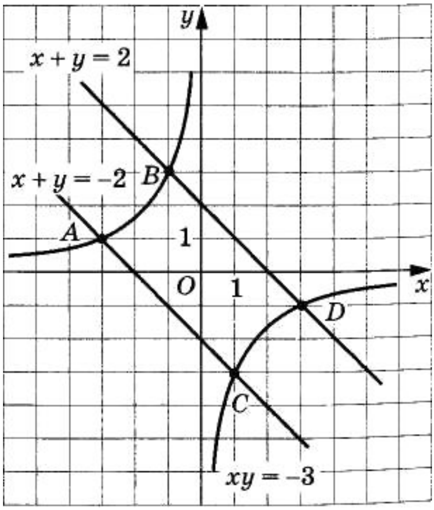

Fig. 3.

Second method. Let us transform the second equation of the system into the form $2xy = -6$ and into the form $-2xy = 6$, and then let us add the first equation of the system $x^2 + y^2 = 10$ first to the equation $2xy = -6$, and then to the equation $-2xy = 6$. We will obtain the system $\begin{cases} x^2 + y^2 + 2xy = 10 - 6 \\ x^2 + y^2 - 2xy = 10 + 6 \end{cases}$, i.e. $\begin{cases} (x+y)^2 = 4 \\ (x-y)^2 = 16 \end{cases}$.

From the first equation we obtain $x + y = 2$ or $x + y = -2$.

From the second equation we obtain $x - y = 4$ or $x - y = -4$.

Examining each of the equations in the first row together with each equation in the second row, we arrive at four systems of linear equations:

$$\begin{cases} x + y = 2, \\ x - y = 4; \end{cases} \quad \begin{cases} x + y = 2, \\ x - y = -4; \end{cases} \quad \begin{cases} x + y = -2, \\ x - y = 4; \end{cases} \quad \begin{cases} x + y = -2, \\ x - y = -4. \end{cases}$$

Solving each of them, we obtain four pairs of numbers:

$$(3, -1), (-1, 3), (1, -3), (-3, 1).$$

This solution is illustrated graphically in Fig. 4. Now we have four lines. The two pairs of lines intersect at the same points that were formed by the intersection of the circle and the hyperbola.

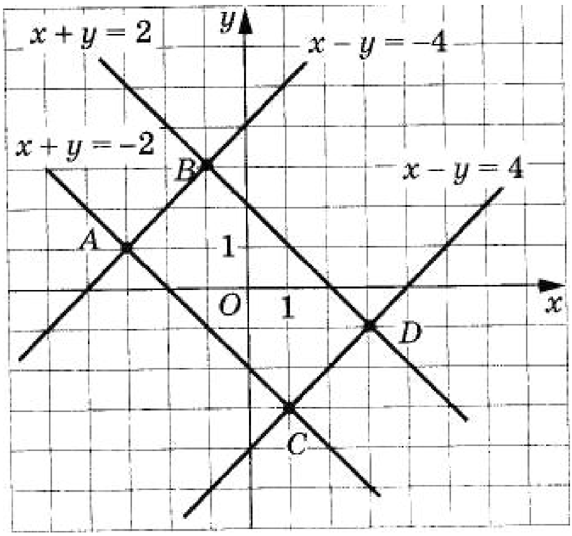

Fig. 4.

Let us thus emphasize one more time that, although in this chapter we are focusing on purely algebraic material, in fact, in the process of teaching, connections with both graphic (analytic) and geometric concepts, ideas, and methods are established whenever possible. In this respect, let us point out another area that is touched on in the section on "Systems of Equations": solving problems with a geometric content by using an algebraic apparatus. In essence, these problems belong to analytic geometry.

It must be noted that the issue of including analytic geometry in school education has been discussed in Russia for nearly an entire century. Although scientists and methodologists have long recognized the importance and significance, as well as the fundamental accessibility, of this material, its actual inclusion in programs for general education was hindered by a variety of circumstances. Problems connected with this topic first appeared in textbooks during the 1970s, a period of reforms in mathematics education, specifically in the textbooks of Makarychev *et al.* In the textbooks of Dorofeev *et al.*, this idea was developed and presented more clearly. Consider the following examples of problems that students were given (Dorofeev, Suvorova *et al.*, 2009a, 2009b):

- Write the equation for the line that passes through the points A $(-1, 2)$ and B $(3, 4)$.
- Write the equation for the line that is: (a) parallel to the line $y = -0.5x + 4$ and passes through the point A $(-6, 5)$; (b) perpendicular to the line $y = -1.5x + 3$ and passes through the point A $(9, 2)$.
- Prove that the three points $(-2, -14)$, $(2, 6)$, and $(3, 11)$ lie on the same line. (Dorofeev and Suvorova, 2009a, pp. 188–189)
- The parabola $y = ax^2 + bx + c$ passes through the points M $(0, 1)$, K $(-1, 0)$, and L $(1, 4)$. Determine whether it passes through the point A $(-4, -5)$.
- The parabola $y = ax^2 + bx + c$ passes through the points M $(0, -2)$, K $(6, 0)$, and L $(3, -4)$. Find the coordinates of its vertex. (2009b, p. 161)

Inequalities. The quantity of material connected with inequalities in basic school is relatively small. All of the textbooks go over the properties of numerical inequalities and the algorithm for solving linear inequalities, which is based on these properties. Systems of linear inequalities are examined with the help of schematic representations of the solution sets of these inequalities on the number line. The properties of numerical inequalities are also used to solve problems involving proofs (for example, to prove such propositions as "the arithmetic mean of two numbers is not less than their geometric mean" or "the half-perimeter of a triangle is greater than any of its sides") and problems on comparing numbers (for example, compare $\sqrt{99} + \sqrt{101}$ and 20).

In addition to solving linear inequalities and systems of linear inequalities, in connection with studying quadratic functions, students solve inequalities of the form $ax^2 + bx + c > 0$, where $a \neq 0$. The essence of the technique used to solve such inequalities lies in the fact that the answer is simply read off a schematically represented graph. No special rules are formulated. Inequalities are also used for solving problems from other sections of the course. Consider the following examples:

1. Find the domain of the expression $\frac{\sqrt{4-2x}}{x+2}$.

2. Find all integer values of m for which the equation

$$4mx^2 + 5x + m = 0$$

has two roots.

3. Find the sum of all positive terms of an arithmetic progression that begins in the following way: 6.3; 5.8; 5.3; … .

Word problems. Considerable attention is traditionally devoted in Russian schools to using the algebraic method to solve word problems. These problems are systematically introduced into the course as the apparatus of equations develops. They are seen as an effective didactic instrument for achieving a number of goals. The main ones among these are:

- Developing the logical thinking of adolescents;
- Demonstrating the possible uses of the algebraic apparatus that is being formed;
- Acquainting students of ages 12–15 with the idea of mathematical modeling on a level that is accessible to them;
- Enriching the educational material with themes that are close, understandable, and interesting to the students, which help to motivate the students.

Although solving word problems, as has already been indicated, figures extensively in the course, this form of activity turns out to be quite difficult and time-consuming for most students. The authors of the modern generation of textbooks, Dorofeev *et al.*, have revised the approach to the content and organization of systems of exercises. They single out as a special form of activity the formulation of equations based on the conditions given in a problem; students acquire experience in formulating different equations on the basis of the same conditions and in determining which of the formulated equations is more convenient for obtaining an answer to the question posed in the problem.

It is important for the students to recognize the necessity of interpreting the numbers obtained by solving an equation or system of equations. Consider the following examples of problems on "Quadratic Equations" for a class working with the textbook of Dorofeev, Suvorova *et al.* (2009a, pp. 120–122).

The students are given three problems:

1. A baking sheet must be made out of a rectangular sheet of tin by cutting out squares in the corners and turning up the edges. The

size of the sheet is 39 cm by 24 cm. What must be the length of the side of one of the squares cut out of the tin sheet in order for the bottom of the baking sheet to have an area of 700 cm^2?

2. The lengths of the sides of the Egyptian triangle are expressed by consecutive positive integers: 3, 4, and 5. Is there any other right triangle whose sides have lengths that are also expressed by consecutive positive integers?
3. A signal rocket is launched at an angle of 45° to the horizon, with an initial velocity of 30 m/s. Its altitude at each moment in time may be calculated approximately using the formula $h = 2 + 21t - 5t^2$. After how many seconds will the rocket reach an altitude of 10 m?

The equation formulated on the basis of the first problem has roots $x_1 = 29.5$, $x_2 = 2$ (x is the length of the side of one of the squares cut out of the tin sheet). The first root is not a solution to the problem, since it is impossible to cut out a square with side 29.5 cm from a tin sheet one of whose sides is 24 cm.

The equation formulated on the basis of the second problem has roots $n_1 = 3$, $n_2 = -1$ (n is the length of the shortest side of the desired triangle). The number -1 does not satisfy the conditions given in the problem, since a length cannot be expressed through a negative number. If $n = 3$, we obtain a triangle with sides 3, 4, and 5. Therefore, the only right triangle whose sides have lengths that are represented by consecutive positive integers is the Egyptian triangle.

The equation in the third problem has roots $t_1 \approx 0.4$, $t_2 \approx 3.8$. In this case, both roots are solutions to the problem. The rocket will be at an altitude of 10 m twice: once on the way up, and once on the way down.

Certain differences in methodological approaches to presenting algebraic material in grades 7–9. The content that forms the foundation for the presentation of algebraic material is the same in all textbooks (as already noted, it is prescribed by the same official document). At the same time, the scientific-methodological principles on which the presentation of educational material in different textbooks is based may differ considerably. This makes it possible for teachers to select that version of the structure of the course which they prefer. Below, we examine, on the basis of the same two series of textbooks by Makarychev *et al.* and Dorofeev *et al.*, the distinctive methodological features that teachers must compare to make a well-informed decision.

First of all, in these two series of textbooks, one finds different attitudes toward the presentation of the theoretical aspects of literal numeration and the theory of equations. Thus, in the textbooks of Makarychev *et al.*, at the very beginning of the presentation of the algebraic material, students are introduced to such basic concepts as identical expressions, identity, equivalent equations, and equivalent transformations of equations. Subsequently, the concept of identity is defined more precisely, in connection with the study of algebraic fractions. The whole subsequent exposition makes use of this terminology, which renders the language of the exposition quite formal and not always well-suited, as we believe and as experience demonstrates, to the intellectual capacities of students of this age.

The authors of this series of textbooks present theory as far as possible in a "rigorous" manner. A substantial number of facts in the textbooks are accompanied by proofs. The authors prefer rigorous computations to plausible-sounding arguments. For example, in the section on "Algebraic Fractions" (Makarychev *et al.*, 2009b), they offer the following proof of the basic property of fractions:

> We know the "basic property of fractions": $\frac{a}{b} = \frac{ac}{bc}$ (a, b, and c are positive integers). Let us prove that this equality holds not only for positive integers but also for any other values of a, b, and c, except $b = 0$ and $c = 0$.
>
> Let $\frac{a}{b} = m$. Then it follows from the definition of a quotient that $a = bm$. Let us multiply both sides of this equality by c: $ac = (bm)c$. Hence $ac = (bc)m$. Since $bc \neq 0$, it follows from the definition of a quotient that $\frac{ac}{bc} = m$. Therefore, $\frac{a}{b} = \frac{ac}{bc}$. (Makarychev *et al.*, 2009b, pp. 7–8).

The series of textbooks by Dorofeev *et al.* takes a different approach to the "rigor" of the exposition. Keeping in mind the significance of mathematics in basic school as a subject aimed first and foremost at general education, the authors, in deciding the question of whether to include this or that proof in a textbook, consider whether it is methodologically indispensable. They deem it necessary to distinguish mathematics itself and the standards of rigor that are accepted in it from the teaching of mathematics and, consequently, the standards of rigor that are appropriate to it. In particular, they carefully take into consideration the age-dependent characteristics of the students, only gradually cultivating their ability to see the necessity of and feel a need for proofs. In keeping with this approach, their textbooks contain all kinds of possible proofs that are accessible to the students' understanding, and whose indispensability the students can appreciate. In the process, the students are introduced to some of the ideas of algebraic proofs — sequences of transformations, algebraic deduction, obtaining a formula by solving a problem in general form, and so on. There are many such proofs in the textbook. In addition, the students learn to prove in the process of solving problems. In presenting the topic of literal numeration, the authors take the following methodological position: the properties of arithmetic operations become the rules of algebra (in essence, axioms, whose number the authors do not attempt to minimize). These are used as a basis on which to formulate rules for transformations that are obvious to the students. This position is initially seen in the seventh-grade course. The same principle of "from numbers to letters" remains in force later on, in the presentation of algebraic fractions. Below, we quote a passage from the textbook by Dorofeev, Suvorova *et al.* (2009a), which corresponds to the passage from the other textbook quoted above:

> The rules for operating with algebraic fractions derive from the rules for operating with ordinary fractions that are known to us from arithmetic. In algebra, these rules become laws that govern the transformations of algebraic fractions. You know the basic property of ordinary fractions, according to which multiplying or dividing the numerator and denominator of a fraction by the same nonzero number yields a fraction that is equal to the given fraction. For example, $\frac{13}{17} = \frac{13 \cdot 4}{17 \cdot 4}$. Algebraic fractions possess a similar property: if the numerator and the denominator of an algebraic fraction are multiplied or divided by the same nonzero polynomial, then the fraction obtained will be equal to the one given. Using letters, this

property is written as follows: $\frac{A}{B} = \frac{A \cdot C}{B \cdot C}$, where $C \neq 0$. (Dorofeev, Suvorova *et al.*, 2009a, pp. 8–9)

As for the terms "identity," "identical expressions," and "equivalent equations," mentioned above, in keeping with the principles just described, they are introduced only in ninth grade, at the final stage of basic education.

There are also differences in the way the material is structured. Thus, in the textbooks by Makarychev *et al.*, discussions on polynomial factorization techniques are "embedded" in material on operating with polynomials. The topic of multiplying a monomial by a polynomial is accompanied by an examination of the technique of factoring by means of collecting like factors; immediately after studying the algorithm for multiplying one polynomial by another, the students are introduced to the factoring-by-grouping method. The textbooks by Dorofeev *et al.* take a different methodological approach. Polynomial factorization is isolated into a separate chapter, which comes after a discussion on operations involving polynomials. Both approaches have their positive aspects. In the first case, this is the simultaneous discussion of forward and backward transformations. In the second case, this is the unified and systematic character of the discussion of an important mathematical problem.

4.1.4 *Examples of test problems*

After finishing basic school, graduates go through a state-mandated final assessment in mathematics in the form of a written exam. The problems on the exam include problems on algebraic material. The exam puts differentiated requirements on student preparation: at the basic and advanced levels. Of 16 problems at the basic level, eight are aimed at testing knowledge in algebra; of five problems at the advanced level, two or three pertain to algebraic material. Thus, the algebraic preparation of students is tested quite thoroughly when they graduate from basic school. Below, we provide examples of problems aimed at testing students' preparation at the basic level on the topics "Algebraic Expressions" and "Equations and Systems of Equations."

For "Algebraic Expressions," exams may include problems aimed at testing students' command of basic concepts, terms, and formulas, as well as their ability to:

- Find the value of an expression with variables when the values of the variables are given;
- Find the domain of a rational expression (integral, fractional), and of elementary expressions containing variables under a radical sign;
- Formulate literal expressions and formulas; carry out computations based on formulas, and express one quantity in a formula in terms of others;
- Carry out transformations of expressions containing powers with natural and integer exponents;
- Transform integral expressions, using the rules for adding, subtracting, and multiplying polynomials, including formulas for $(a \pm b)^2$ and $(a - b)(a + b)$;
- Factor polynomials by factoring out common factors and by using formulas for short multiplication; factor quadratic trinomials;
- Reduce fractions and transform simple fractional expressions; carry out transformations of numeric expressions containing square roots. (Kuznetsova *et al.*, 2009, pp. 43–48)

Examples of problems are given below (some of them require a short answer, some are multiple-choice questions, and some require students to match questions with answers) (pp. 43–48):

1. Find the value of the expression $1.5x^3 - 0.8x$ for $x = -1$.
2. Find the value of the expression $\frac{1-\sqrt{a}}{\sqrt{b}}$ for $a = 0.64$ and $b = 0.09$.
3. Given the expressions (1) $\frac{a+3}{a}$, (2) $\frac{a}{a+3}$, and (3) $\frac{a+\frac{3}{a}}{3}$, which of them are not defined for $a = 0$?

   (1) Only 1 (2) Only 3 (3) 1 and 3 (4) 1, 2, and 3

4. For which of the following values of x is the expression $\sqrt{12 + 3x}$ not defined?

   (1) $x = 0$ (2) $x = -6$ (3) $x = -1$ (4) $x = -4$

5. The distance in meters to the epicenter of a storm can be computed approximately by using the formula $s = 330t$, where

t is the number of seconds that have passed between a stroke of lightning and a clap of thunder. Determine the approximate distance of an observer from the epicenter of the storm if $t = 12$. Give the answer in kilometers, rounding it off to an integer.

6. A car uses a L of gasoline to drive 100 km. How many liters of gasoline will be needed to drive 37 km?

   (1) $\frac{a \cdot 37}{100}$ L (2) $\frac{100 \cdot 37}{a}$ L (3) $\frac{a \cdot 100}{37}$ L (4) $\frac{a}{37 \cdot 100}$ L

7. The area of a circle with diameter d is computed using the formula $S = \frac{\pi d^2}{4}$. Use this formula to define diameter d.

   (1) $d = \frac{4S}{\pi}$ (2) $d = \sqrt{\frac{4S}{\pi}}$ (3) $d = \sqrt{\frac{\pi S}{4}}$ (4) $d = \sqrt{\frac{\pi}{4S}}$

8. For each expression in the top row, indicate the expression in the bottom row that is equal to it.

   (A) $a^{-8} \cdot a^2$ (B) $\frac{a^{-8}}{a^2}$ (C) $(a^{-8})^2$

   (1) a^{-16} (2) a^{-10} (3) a^{-6} (4) a^{-4}

9. Express the value of the expression $(6 \cdot 10^{-3})^2$ in the form of a decimal fraction.

10. In which case is the expression transformed into an equal expression?

    (1) $3(x - y) = 3x - y$ (3) $(x - y)^2 = x^2 - y^2$

    (2) $(3 + x)(x - 3) = 9 - x^2$ (4) $(x + 3)^2 = x^2 + 6x + 9$

11. Simplify the expression $6x + 3(x - 1)^2$.

    (1) $3x^2 + 3$ (3) $9x^2 - 6x + 9$

    (2) $3x^2 + 1$ (4) $3x^2 + 6a - 3$

12. Reduce the fraction $\frac{ab^2 - 2ab}{2ab}$.

    (1) ab^2 (2) $\frac{b-2}{2}$ (3) $b^2 - a$ (4) $b - 1$

13. Indicate the expression that is identical to the fraction $\frac{a-c}{b-c}$.

    (1) $\frac{c-a}{b-c}$ (2) $\frac{a-c}{c-b}$ (3) $\frac{c-a}{c-b}$ (4) $-\frac{c-a}{c-b}$

14. Simplify the expression $\frac{2m - 4m^2}{m+1} \div \frac{2m^2}{m+1}$.

15. Find the value of the expression $2\sqrt{13} \cdot \sqrt{2} \cdot 5\sqrt{26}$.

As experience shows, students are relatively good at finding values of expressions with variables when the value of the variable is given, at formulating a literal expression based on the conditions given in a problem, and at expressing one quantity in a formula in terms of others.

The most difficult types of problems in this set are problems that test students' grasp of the concept of the domain of a rational expression and problems that involve operations with algebraic fractions (even though the demands made on the students are quite modest, as can be seen from the problems reproduced above).

For "Equations and System of Equations," exams may include problems aimed at testing students' command of basic concepts, terms, and formulas, as well as their ability to:

- Solve linear and quadratic equations, as well as equations that can be reduced to linear and quadratic equations, by means of simple transformations; solve integral equations by relying on the fact that a product is equal to zero; solve simple linear-fractional equations;
- Carry out elementary investigations of quadratic equations (to establish whether an equation has roots, and if so, how many);
- Know and understand the following terms: "equation with two variables," "solving equations with two variables," and "the graph of an equation with two variables"; understand the graphic interpretation of an equation with two variables, and of a system of equations with two variables;
- Solve systems of two linear equations with two variables and simple systems of two equations of which one is quadratic;
- Formulate an equation with one variable or a system of equations with two variables based on the conditions given in a word problem.

Examples of possible problems are given below (pp. 49–54):

1. Solve the equation $3 - 2x = 6 - 4(x + 2)$.
2. Solve the equation $\frac{x}{2} - 3 = \frac{x}{5}$.
3. Find the roots of the equation $3x^2 + x = 0$.
4. Indicate how many roots each equation has:

   (A) $(x+1)^2 = 0$ (B) $x^2+1 = 0$ (C) $x^2+x = 0$ (D) $x^2-x = 0$

   (1) One root (2) Two roots (3) No roots

5. Which of the following equations has irrational roots?

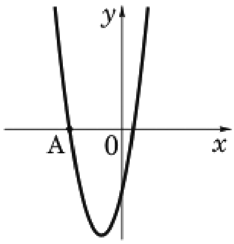

Fig. 5.

(1) $x^2 - 3x - 4 = 0$ (3) $x^2 - 4x + 5 = 0$

(2) $x^2 - 4x - 3 = 0$ (4) $x^2 - 4x + 4 = 0$

6. Find the roots of the equation $(2x - 5)(2 + x) = 0$.
7. Figure 5 shows the graph of the function $y = 2x^2 + 3x - 2$. Determine the x coordinate of the point A.
8. Solutions to the system of equations $\begin{cases} x + y = 2 \\ xy = -15 \end{cases}$ are:

(1) (5, −3), (−5, 3) (3) (5, −3), (−3, 5)

(2) (−5, 7), (3, −1) (4) (−5, 7), (5, −7)

9. In which quadrant of the coordinate plane does the point of intersection of the lines $2x - 3y = 1$ and $3x + y = 7$ lie?

(1) I (2) II (3) III (4) IV

10. In the coordinate plane (Fig. 6) points P and Q are marked and a line is drawn through them. Which equation defines this line?

(1) $x + y = 16$ (2) $x + y = 26$ (3) $x - y = 4$ (4) $x - y = 5$

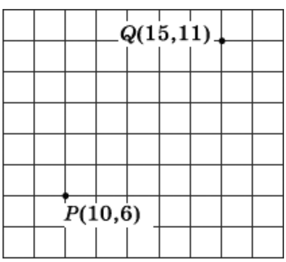

Fig. 6.

11. Read the following problem:

    The distance between two marinas is 17 km. A boat sailed from one marina to the other and back in 6 h. Find the boat's own speed if the speed of the river's current is 2 km/h.
    Use the letter x to designate the boat's own speed (in km/h) and formulate an equation based on the conditions given in the problem. Which of the following is the right answer?

    (1) $\frac{17}{x+2} + \frac{17}{x-2} = 6$ (3) $\frac{17}{x+2} = \frac{17}{x-2} - 6$
    (2) $\frac{x+2}{17} + \frac{x-2}{17} = 6$ (4) $17(x+2) + 17(x-2) = 6$

Experience shows that students on the whole are good at solving linear and quadratic equations. However, the number of correct answers goes down if an equation has fractional coefficients (for example, $\frac{1}{3}x^2 + x - 6 = 0$). In general, whenever in any context students must work with fractions, they begin having difficulties. Many students have difficulty solving a basic, standard problem that is present in all textbooks: compute the coordinates of the point of intersection of two straight lines by solving a system of two linear equations with two variables. The greatest difficulty for students then arises when they must formulate an equation based on the conditions given in a word problem.

We will now illustrate the requirements that must be met by the algebraic preparation of students at the advanced level.

For "Algebraic Expressions," exams may include problems aimed at testing students' command of the following skills (Kuznetsova *et al.*, 2009, p. 72):

- Factoring polynomials using different methods;
- Carrying out many-step transformations of rational expressions using a wide array of studied algorithms;
- Carrying out transformations of expressions that contain powers with integer exponents, and square roots;
- Carrying out transformations to solve various mathematical problems (such as problems on finding maxima and minima).

Examples of possible problems are (the solutions to all of these problems must be written out, and their precision and completeness have a substantial influence on the grade):

1. Factor the polynomial $c^2a - a - c^2 + 1$.
2. Reduce the fraction $\frac{4a^2-9a+2}{1-4a+x-4ax}$.
3. Simplify the expression $\left(\frac{b-3}{b^2-2b-3} - \frac{b}{b^2+2b+1}\right) \div \frac{1}{(5b+5)^2}$.
4. Show that for any value of n the expression $\frac{5^{n+1}+5^{n-1}}{2\cdot 5^n}$ has the same value.
5. Find the value of the expression $\sqrt{(2\sqrt{7}-5)^2} + \sqrt{(2\sqrt{7}-6)^2}$.
6. For what values of the variable does the following expression is not defined? $1 - \frac{1}{1-\frac{a}{1-\frac{1}{a+1}}}$
7. Prove the following identity:
   $(x+1)(x+2)(x+3)(x+4) + 1 = (x^2+5x+5)^2$.
8. Prove that there are no values of a and b for which the value of the expression $5a^2+3b^2+20a-12b+34$ is equal to zero. (pp. 72–73)

For "Equations and Systems of Equations," the exam may include problems aimed at testing students' command of the following skills:

- Solving integral and fractional equations with one variable by means of algebraic transformations and such techniques as factorization and variable substitution;
- Solving systems of linear equations and systems containing nonlinear equations by means of substitution and addition; also using certain special techniques;
- Carrying out investigations of equations and systems of equations containing letter coefficients, in particular by relying on graphic representations;
- Solving word problems, including working with models in which the number of variables is greater than the number of equations.

Examples of problems are given below (again, all of them require full written solutions).

1. Find the roots of the following equation: $2x^4 - 17x^2 - 9 = 0$.
2. Solve the following equation: $\frac{x}{3x+2} + \frac{5}{3x-2} = \frac{3x^2+6x}{4-9x^2}$.
3. Solve the following equation: $(x^2-3x-1)^2 + 2x(x-3) = 1$.

4. Solve the following system of equations:
$$\begin{cases} 2(x-y)-3(x+y)=2x-6y \\ \frac{x+y}{2}-\frac{x-y}{5}=\frac{2x}{5}-2 \end{cases}.$$
5. Solve the following system of equations:
$$\begin{cases} 3(x+y)+xy=-14 \\ x+y-xy=6 \end{cases}.$$
6. Solve the following system of equations: $\begin{cases} 2x-3y=-7 \\ 4x+5y=14 \\ x^2+y^2=10 \end{cases}$.
7. Given the system of equations $\begin{cases} \frac{x}{3}-\frac{z}{4}+\frac{y}{12}=1 \\ \frac{y}{5}+\frac{x}{10}+\frac{z}{3}=1 \end{cases}$, find the sum $x+y+z$.
8. Find all negative values of m for which the system of equations $\begin{cases} x^2+y^2=m^2 \\ x+y=1 \end{cases}$ has no solutions.

Solve the following problems (9–11):

9. Three candidates were running for the position of team captain: Nikolayev, Okunev, and Petrov. Petrov got three times as many votes as Nikolayev, while Okunev got two times fewer votes than Nikolayev and Petrov combined. What percentage of the votes was cast for the winner?
10. A student was planning to live for a certain number of days on 600 rubles. During each of the first three days, he spent the sum he had planned on spending each day; then he increased his daily expenditures by 20 rubles. As a result, by two days before the end of the period he had already spent 580 rubles. How much money had the student planned on spending each day?
11. By mixing one salt solution, whose concentration is 40%, with another solution of the same salt, whose concentration is 48%, we obtain a solution with a concentration of 42%. In what proportions were the first and second solutions mixed? (pp. 75–77)

4.2 *High School (grades 10–11; students aged 16–17)*

4.2.1 *An overview*

As indicated above, after finishing basic school each student is offered a choice: to study mathematics in high school (grades 10–11) at the basic or advanced ("profile") level. The students must make this choice in accordance with the life goals they set for themselves at the moment. The division of education into basic and advanced levels (which exists in all subjects, not just in mathematics) makes it possible for the students to concentrate their efforts on the more intensive study of that specific range of subjects which will, in the future, be connected with their professional activity or are simply more interesting to them. Thus, if the main goal of studying mathematics at the basic level is to facilitate the general cultural development of the students, then at the advanced level, in addition to this goal, the study of mathematics must provide the students with the possibility of entering those departments of institutions of higher learning in which mathematics is one of the main subjects and of continuing their professional education there.

Precisely this difference in the aims of high school mathematics education was at one time one of the main factors that determined the differences in the content of mathematics education and in the graduation requirements in mathematics that were set down by the Standard. Precisely this difference determines the difference in approaches to presenting material in mathematics textbooks in high schools at the basic and advanced levels.

As in basic school, different series of textbooks are used for studying mathematics at each educational level (basic or advanced) in high school; all of these textbooks meet the Standard's requirements, but their methodological approaches vary. It should be noted that prior to the official introduction of furcation at the high school stage (established in 1998), mathematics in all schools (except for classes with an advanced course of study in mathematics, which used their own original programs) was officially studied on one level (in the early 1990s, classes oriented toward the humanities also began to appear). Since approximately 1.5% of students in Russia select an advanced course in mathematics, and since such classes are usually taught by authoritative and very highly qualified teachers who use their own original curricula and unique methodologies, which cannot be generalized and applied on a mass scale, this layer of Russian mathematics education will not be addressed in this chapter.

The content of the contemporary basic course in mathematics corresponds to the content of the general course that was previously in use, with minor changes in the direction of reducing the amount of educational material and simplifying the requirements for the level of its assimilation. This makes it possible to employ all existing educational toolkits in teaching this course.

We should point out that mathematics as presented in the basic course, which is intended for the majority of Russian schoolchildren, is conceived of as being just one element of general education. However, although we have been teaching mathematics to all students at the high school level for many decades, ever since secondary education was made mandatory, no traditions of teaching this subject as part of general education have yet taken root in Russia.

As for the advanced course in mathematics, one of the main objectives that it is meant to address — to teach mathematics in accordance with the modern conception of school mathematics education and, at the same time, specifically, to provide students with the possibility of entering a college — could not have been achieved using existing textbooks, in our view, and required fundamentally new developments in the field of educational literature.

To illustrate the contemporary state of mathematics teaching at the basic and advanced levels, we will look at two educational-methodological sequences. The first consists of the textbook for grades 10–11 by Kolmogorov *et al.* (2007), whose original version came out in the 1970s, at the same time as the already-mentioned textbooks by Vilenkin *et al.* and Makarychev *et al.*, and traditionally considered their continuation. Since that time, these textbooks have been substantially revised. At present, they are the most widely used textbooks in Russian schools.

The second series of textbooks, by Dorofeev, Kuznetsova, and Sedova (2003, 2008), Dorofeev and Sedova (2007), and Dorofeev, Sedova, and Troitskaya (2010), which was written with the participation of some of the authors of this chapter, may be seen as a development of the ideas contained in the textbooks by Dorofeev, Sharygin *et al.* and Dorofeev, Suvorova *et al.* These are new textbooks for students who wish to acquire a deeper education in mathematics.

These two groups of textbooks, in our view, reflect a fundamental difference between two conceptions of basic and advanced courses in mathematics.

4.2.2 *The study of algebraic expressions in grades 10–11*

The content of the material pertaining to the study of algebraic expressions in high school, like the other sections of the course in mathematics, is prescribed by the Standard. It should be noted that certain topics are listed in the Standard in italics; these topics must be included in the curriculum but are not part of the final attestation. Also, the Standard does not require that the high school curriculum include a section specifically devoted to numbers. Issues connected with expanding the concept of number thus belong to the algebraic part of the curriculum.

As can be seen from the passages from the Standard cited above, the content that pertains to the study of algebraic expressions differs substantially in the basic and advanced courses. Their common part is connected with the study of roots of degree n, powers with rational exponents, and the logarithm of a number. In these sections, students receive virtually the same set of theoretical facts, so the main difference is in the depth of their assimilation of this material.

As an illustration of this difference, consider how the students learn the topic "Roots of the nth degree and their properties."

In the textbook by Kolmogorov *et al.* (2007), the emphasis is on learning definitions and algorithms. Thus, in studying this topic, students must assimilate certain techniques for transforming algebraic expressions. At the mandatory level, they must learn how to solve the following types of problems:

- Move a factor outside the radical sign ($a > 0$, $b > 0$):
  (a) $\sqrt[6]{64a^8b^{11}}$;
  (b) $\sqrt[5]{-128a^7}$.
- Move a factor inside the radical sign ($a > 0$, $b > 0$):
  (a) $-b\sqrt[4]{3}$; (b) $ab\sqrt[8]{\frac{5b^3}{a^7}}$. (Kolmogorov *et al.*, 2007, p. 205)

A somewhat higher level is illustrated by the following problem:

- Put the following expression in the form of a fraction whose denominator does not contain a radical: (a) $\frac{1}{\sqrt[3]{2}-\sqrt[3]{3}}$; (b) $\frac{2}{a-\sqrt[3]{b}}$. (Kolmogorov *et al.*, 2007, p. 206)

The corresponding technique, as we will see below, is used in solving irrational equations.

In the advanced textbooks of Dorofeev, Kuznetsova, and Sedova deliberately learning to carry out elementary algorithms is not an end in itself. Transformations involving radicals as a rule play a secondary role and have the character of technical work, which must be carried out in the process of solving more substantive problems.

For comparison, consider several problems on the topic examined above from the problem book of Dorofeev, Sedova, and Troitskaya (2010). Of course, as in Kolmogorov *et al.*'s (2007) textbook, this problem book includes problems that involve elementary simplifications of expressions with radicals. But this problem book also examines the opposite problem: under what conditions (constraints on variables) is an already-transformed expression equal to the one given? (Dorofeev, Sedova, and Troitskaya, 2010, p. 16).

For what x and y is the expression $(y-5)\sqrt{\frac{x-15}{y-5}}$ equal to:
(a) $\sqrt{(x-15)(y-5)}$; (b) $-\sqrt{(x-15)(y-5)}$; (c) $\sqrt{(15-x)(y-5)}$?

Considerable emphasis is placed on the understanding of the connection between this new concept and other areas of mathematics. Thus, for example, for practical purposes, a student has no need to think about the fact that the root of a positive integer cannot be anything other than a positive integer or irrational number, but the future mathematician must understand this.

Students may recall how, in basic school, they proved by contradiction that certain roots are irrational: for example, that the number $\sqrt{2}$ is irrational. Now they possess an instrument that makes it possible to prove at once that all numbers of this form are irrational (the textbook discusses how the rational roots of the polynomial $f(x) = x^n - k$ can only be divisors of the number k, i.e. integers). Therefore, a problem that requires students to prove this fact is included among the problems in this section.

Problems involving transformations of expressions with radicals by means of multiplying them by a "conjugate factor" are traditionally widespread (Kolmogorov *et al.*, 2007, p. 206). The advanced problem book offers a more substantive problem:

> Is the following function monotonic? $y = \sqrt{x+1} - \sqrt{x-2}$. (Dorofeev, Sedova, and Troitskaya, 2010, p. 15)

This problem is solved precisely by multiplying by "conjugate factors": after the corresponding transformation, we obtain another expression for the same function, $y = \frac{3}{\sqrt{x-2}+\sqrt{x+1}}$, from which it can be seen that this function is a decreasing function (the numerator of the fraction is a positive constant, while the denominator is increasing). Of course, by including one problem within another in this way, we always obtain a problem that allows for several solutions; and indeed the student has the right to dispense with multiplying by a "conjugate factor," which, however, does not seem worrisome. The problem book also contains problems involving proofs of formulas with double radicals:

> (a) $\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$;
> (b) $\sqrt{a-\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} - \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$.

This problem constitutes both an exercise in transforming expressions with radicals and a certain addition to students' algebraic arsenal: of course, they do not need to memorize this formula, but it is important for understanding the fact that sometimes (if the number $a^2 - b > 0$ is a perfect square and $a > 0$) one can eliminate a complicated radical by turning it into the sum of two simple radicals; this technique is sometimes used to solve biquadratic equations.

As an example of an even more difficult assignment that involves a proof, consider the following problem:

> Prove that $\frac{5}{3} < \frac{1}{\sqrt{26}} + \frac{1}{\sqrt{27}} + \cdots + \frac{1}{\sqrt{35}} < 2$. (Dorofeev, Sedova, and Troitskaya, 2010, p. 19)

Here, too, the solution cannot be reduced to an operation involving radicals. The student must note that

$$\frac{10}{\sqrt{35}} < \frac{1}{\sqrt{26}} + \frac{1}{\sqrt{27}} + \cdots + \frac{1}{\sqrt{35}} < \frac{10}{\sqrt{26}},$$

since each of the given ratios, beginning with the second one, is less than that preceding it — after which it is not difficult to see that $\frac{5}{3} < \frac{10}{\sqrt{35}}$ and $\frac{10}{\sqrt{26}} < 2$.

This section also covers the topic "Divisibility." At this point, we must explain in greater detail our understanding of what an advanced course in mathematics in high school must achieve, and the fundamental difference between such a course and a course that results simply from the addition of certain topics to the basic course in mathematics (unfortunately, there is a common but — in our view — erroneous opinion that this latter type of course is just what constitutes an advanced course in mathematics).

As an example, let us consider precisely the topic "Divisibility." Why was this topic included in the content of the curriculum? We can point to many reasons for this, but the main, most "conceptual" one apparently had to do with the fact that issues connected with divisibility are far more important for mathematics than, say, solving irrational equations; in other words, this topic *brings the content of the school course closer to real mathematics*. In particular, knowledge of this material makes it possible, in studying the topic "Polynomials with one variable" (another topic that distinguishes the advanced course from the basic, examined in greater detail below), to study questions connected with the rational roots of polynomials with integer coefficients, i.e. to solve a broader range of higher-degree equations, and in turn to make use of this knowledge in studying rational and irrational numbers, and so on.

We might also mention that in solving problems pertaining to these topics, students learn to use not so much the algorithms for solving some narrow class of problems as the methods and techniques of mathematical activity in general.

At present, problems connected with divisibility are generally thought of as belonging to the category of so-called Olympiad problems; but this is so only because in the existing course in mathematics, this content is covered effectively only in grades 5–6 and essentially has a narrowly directed aim — to develop certain well-defined arithmetic abilities and skills.

The stylistic aspect of this topic's presentation is determined first and foremost by the objectives associated with studying mathematics in basic school, where it is by no means assumed that most students will take the advanced course in mathematics in high school. It is also limited objectively by the age-dependent characteristics of the students — the highly concrete nature of their thinking, which makes it difficult for them to interact with abstract objects, and with letters in particular, because of their insufficiently developed capacity for making theoretical generalizations, and for understanding the essence of proofs and their role in mathematics; because of their lack of any felt need to prove propositions "in the general form" when confronted with conclusive concrete examples; and so on.

However, these traits are no longer characteristic, by and large, of 16–17-year-old teenagers, especially those who have gone through three more years of schooling in a different style that is more in harmony with the essence of mathematics and, above all, those who have chosen an advanced course of study, designed essentially for the formation of the country's "technical–scientific elite."

This position became more or less central in the general approach of the textbooks and problem books of Dorofeev, Kuznetsova, Sedova, and Troitskaya. "Divisibility" is the first topic presented in these textbooks, mainly with a view of providing continuity with the content of the basic school curriculum, but also in consideration of the objective simplicity of its content and its proximity to experiences that students already have. The difficulties with its assimilation (both on the level of theory and, to an even greater degree, on the level of exercises) are connected with a purely psychological barrier: the unfamiliarity of the mathematical activity that corresponds to the content of the material. In particular, in treating this topic, the authors of this textbook use material that is quite simple to form and develop the students' ability to formulate proofs; this ability, as is well known, is one of the most significant weak spots in the mathematical preparation of students. In doing so, the authors have not deemed it necessary to fill in all the logical gaps that have been left by the study of divisibility in basic school; for example, they do not consider it necessary even to prove the criteria for divisibility by 3 and 9 in the general case. Thus, for example, they present the proof of criteria for divisibility by 11 in basic-school fashion, based on presenting an example, the generality of which is obvious to any mathematician and must be equally obvious to any student. A formal proof of this fact requires only a complicated mathematical "ornament" and, apart from logical rigor (which in this instance seems superfluous), adds nothing to the mathematical content of the argument or, most importantly, to the basic problem of developing the students' mathematical thinking. Moreover, the very fact that students have understood the generality of an example that conclusively demonstrates the mechanism of a *potential* formal proof constitutes an important contribution to their mathematical thinking, promoting those peculiar features of thought which are characteristic of mathematicians and necessary for assimilating mathematics.

Let us note that the concept of logical thinking, the thinking that is used in mathematics and to an even greater degree by representatives of other sciences, is substantially broader than that of *deductive* thinking — a fact that many representatives of the methodological disciplines and practicing teachers sometimes forget, losing or substantially weakening the *productive* component of thinking by doing so.

Everything that has been said above pertains, of course, not just to the topic "Divisibility," but illuminates the way in which an advanced course in mathematics must differ from the basic course, what the general principles governing the design of the advanced course must be, and what approach must be used, in our view, to solve the corresponding methodological problems.

Let us examine concrete problems for students that reflect the authors' approach to the topic "Divisibility" in the aforementioned textbooks by Dorofeev *et al.*

1. Prove or refute the following statements:

   (a) All even numbers are composite; (b) if an even number is divisible by 15, then it is divisible by 6; (c) if an even number is divisible by 15, then it is divisible by 20.

2. Prove that:

   (a) $3^{2003} + 3^{2004} + 3^{2006}$ is divisible by 31;
   (b) $20^{186} + 18^{253}$ is divisible by 19.

3. Find the remainder after dividing:

   (a) $6n + 5$ (n — integer) by 3;
   (b) $6n + 5$ (n — positive integer, greater than 1) by n;
   (c) 2^{2005} by 7.

4. Which of the progressions

   5, 8, 11,...; 4, 7, 10,...; 6, 9, 12,...

   contains the number $11 \cdot 38^{20} - 4 \cdot 25^{10}$? (Dorofeev, Kuznetsova, Sedova, and Okhtemenko, 2004, p. 38)

As we can see, there is no general rule, no algorithm, and no general ability for solving these problems except one: the ability to reason. Not for nothing was the topic "Divisibility" traditionally a favorite topic for problems on college entrance exams, at a time when there was no Uniform State Exam.

Clearly, despite the simplicity of the formulations of these problems, the basic level of preparation is not enough to solve them — and this has to do not with new, additional criteria for divisibility (for example, criteria for divisibility by 11), which may or may not be present in the textbook of the advanced course; or with new concepts and theorems that the Standard prescribes for the advanced level (such as "Congruences"). It has to do with the depth with which those concepts are assimilated, which are already known to all graduates of basic schools. The Standard does not stipulate the study of any ready-made algorithms for solving such problems; rather, what is required of graduates here is the ability to engage in mathematical reasoning in nonstandard situations.

With regard to significant differences between the content of the basic and advanced courses, we should also look at the topic "Polynomials," which is studied in advanced classes. The main purpose of this topic, according to the Standard for advanced schools, is to improve the general mathematical preparation of the students, and to help them learn simple and effective techniques for solving problems, especially algebraic equations.

Without presenting any fundamental difficulties, the study of polynomials gives students the possibility of solving many problems that belong to all other parts of the course. In particular, this theoretical content can be effectively used in solving problems connected with prime and composite numbers, while the ability to find the rational roots of polynomials with integer coefficients allows the students not to be too afraid of cubic equations and higher-degree equations — in many cases, to stop relying on the *art* of grouping (i.e. heuristic techniques) and to make use instead of the algorithmic methods of the theory of polynomials; to simplify standard proofs; and so on.

It should also be noted that the study of polynomials provides a fitting conclusion to the generalization of the concept of number, while the parallelism between the theory of polynomial factorization and the outwardly very different theory of integer factorization, unexpected for the students, is important from a general educational and general cultural point of view.

Let us consider some examples pertaining to this topic. The following problem provides a useful illustration:

Is the expression $\frac{1}{x^2+1}$ a polynomial?

This problem calls for a well-founded answer. Naturally, the main point here is for students to grasp the concept of a polynomial in a substantive sense, and therefore excessive attention to formalities in defining this concept is unlikely to be fruitful. Attempts to give a logically impeccable definition of a polynomial will merely lead to formulations with which probably not even all professional mathematicians are familiar. On the other hand, in trying to identify a polynomial among other expressions, a logically developed student must understand that, strictly speaking, this cannot be judged merely by the external form of an expression. Thus, for example, the expression $\frac{1}{x^2+1}$ is not a polynomial not because there do not appear to be any algebraic transformations that can be used to put it into the appropriate form, but because there actually are no such transformations. Indeed, supposing that the given expression is a polynomial, then from the equality $\frac{1}{x^2+1} = f(x)$, where f is a polynomial of degree n, there would follow the equality $1 = (x^2 + 1) f(x)$; but this equality is impossible, since its left-hand and right-hand sides have different degrees.

The algorithm for searching for rational roots must be worked on until it becomes a familiar skill. Students must not experience difficulties when they encounter problems of the following type:

- Find all the roots of the following polynomial:
  $3x^6 - 14x^5 + 28x^3 - 32x^2 - 16x + 16 = 0$.
- Factor the polynomial $f(x) = 3x^4 - 2x^3 - 9x^2 + 4$ into linear factors.

When studying divisibility and division with a remainder, there is no need, for most polynomials, to list completely, much less to memorize, the criteria of divisibility. On the contrary, it is far more useful to emphasize to the students that many properties of the divisibility of integers that are known to them are present in the divisibility of polynomials as well. But the students must also be asked to prove these properties (or some part of them) on their own, and in the process of formulating these proofs they will conclude for themselves that the arguments differ only because of their terminology and symbolism.

Another theme that distinguishes the advanced course from the basic one is connected with the concept of a symmetric polynomial. In our view, students who have chosen the advanced course could have already learned at the basic-school level how to solve various problems that require only identity transformations aimed, to put it in a lofty way, at expressing any symmetric polynomials in terms of elementary symmetric ones. Note that various ordinary identity transformations effectively constitute the central content of algebra in basic school, but are often lacking in ideas and aimed mainly at simplifying expressions. Such a situation even compromises mathematics to some degree in the eyes of the students: it almost seems as if someone had deliberately complicated simple expressions to create difficulties for students. Meanwhile, the concept of the symmetric polynomial makes it possible to introduce substantive problems of another type. For example, expressing the sum $x^3 + y^3 + z^3$ through the elementary symmetric polynomials

$$x^3 + y^3 + z^3 = (x + y + z)^3 - 3(x + y + z)(xy + xz + yz) + 3xyz$$

makes it possible to solve the most varied problems. Thus, using the identity just given, it is not difficult to deduce that the last three digits of the number $423^3 + 255^3 + 322^3 - 423 \cdot 255 \cdot 272$ are zeroes, since $x^3 + y^3 + z^3 - 3xyz$ is divisible by $x + y + z$. From the same identity follows the inequality $x^3+y^3+z^3-3xyz \geq 0$, i.e. in essence an inequality between the arithmetic mean and the geometric mean.

4.2.3 *Equations and inequalities in the basic and advanced courses in mathematics in grades 10–11*

As in the previous section, the sections on "Equations and Inequalities" in the basic and advanced courses have a certain shared component, which is mainly connected with standard techniques for solving irrational equations.

Thus, in the textbook by Kolmogorov *et al.* (2007), the technique for solving irrational equations in essence amounts to the method of squaring both parts of an equation and subsequently checking for roots to exclude extraneous ones. This method is quite legitimately employed for all equations with radicals. Consider the following example:

Solve the equation $\sqrt{x-6} = \sqrt{4-x}$.

As an illustration, let us quote a passage from the textbook that pertains to the solution of this equation: "Squaring both sides of this equation, we obtain $x - 6 = 4 - x$, $2x = 10$, $x = 5$. By substituting, we conclude that the number 5 is not a root of the given equation. Therefore, the equation has no solutions" (Kolmogorov *et al.*, 2007, p. 207).

In the advanced course, somewhat more attention is devoted to this topic, since the aim here is not merely to make students learn certain simple algorithms, but first and foremost, as has already been said, to develop their awareness of underlying connections between different parts of the course. Therefore, the standard algorithm, which remains a kind of "magic wand," gradually gives way to considerably more tricky maneuvers, which make it possible to shorten solutions substantially and even to solve a certain range of problems mentally. These techniques are based on the investigation of the domains of the left-hand and right-hand sides of the equation, their ranges, and the properties of the functions that enter into the equation.

Let us use an example to clarify what has just been said. Thus, in the advanced course, the equation given above does not need to be solved straightforwardly. The students are already sufficiently prepared to "see" that the domains of the left-hand and right-hand sides of the equation do not intersect, so that there simply is no place for roots in this equation.

Furthermore, in the advanced course, in addition to equations, students solve irrational inequalities. The following examples show the level of difficulty of these problems and the variety of techniques used in solving them:

- Solve the equation $\sqrt[3]{x+1} + \sqrt[3]{x+2} + \sqrt[3]{x+3} = 0$.
- Solve the equation $4(\sqrt{x^2-1})^3 - 3x^2\sqrt{x^2-1} = x^3$.
- Solve the inequality $\sqrt[3]{2x - x\sqrt{x} - 1} + \sqrt{x} + \sqrt[3]{1-2x} \leq 0$. (Dorofeev, Sedova, and Troitskaya, 2010, pp. 44, 47, 51)

The first of these problems allows for a mental solution based on the properties of the monotonic function — the left-hand side of the equation is an increasing function, and therefore it assumes the value 0 at no more than one point. By trial and error, it is possible to determine that $x = -2$ is a root of this equation. Thus, this is the only solution to this equation.

The solution to the second of these problems becomes noticeably more simple if one uses the trigonometric substitution $x = \frac{1}{\cos t}$. As for the third problem, a "trained eye" will see that substituting

$$\sqrt[3]{1-2x} = a, \sqrt{x} = b,$$

leads to a simpler inequality, $a + b \leq \sqrt[3]{a^3 + b^3}$. After raising the inequality to the third power and making the appropriate identity transformations, we obtain an expression of the form $f(a) \leq 0$, where $f(a)$ is a second-degree polynomial whose roots relative to the variable a are obvious: $a = 0$ and $a = -b$. After this, we can return to the original variable and, for example, use the interval method.

4.2.4 *The final attestation in algebra for 11th graders*

Upon completing their mathematics education, all graduates of secondary general education schools in Russia must take the Uniform State Exam (USE). Without going into a detailed discussion on the structure and aims of the USE here,[1] we will confine ourselves to describing several problems from the USE just on the two topics examined in this chapter.

The transformation of algebraic expressions. The "difficult" part (C) of the exam contains no problems specifically on this topic, although the solutions to problems in other topics require sufficient proficiency in carrying out transformations of algebraic expressions. The following problem is typical of an easier section:

> Find the value of the expression $\frac{(2\sqrt{7})^2}{14}$.

As can be seen, the students' ability to operate with roots of degree n is tested on quite primitive examples. The problems have a purely technical character.

Equations and inequalities. The problems in this "easy" section on a different topic also presuppose command of the standard algorithm. The knowledge provided by the basic course is sufficient to solve them. Consider the following example:

> Find the root of the equation $\sqrt{-72 - 17x} = -x$. If the equation has more than one root, indicate the lesser of them.

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[1] Editorial note: For a more detailed treatment of the USE, see Chapter 8 of this volume.

The following problem may serve as an example from a more difficult section of the exam (in this section, this problem is about average in difficulty):

Solve the inequality $\sqrt{7-x} < \frac{\sqrt{x^3-6x^2+14x-7}}{\sqrt{x-1}}$.

Formally, this problem is only for advanced-course graduates, since the basic course does not address solving irrational inequalities. Although the probability that this problem will be solved by basic-level graduates is by no means zero — if it does not scare them off immediately, this inequality can be solved by graduates who have completed the basic program (it is not so difficult to see that one must solve the inequality $8x - x^2 - 7 < x^3 - 6x^2 + 14x - 7$ on the interval $(0, 7]$) — the general tendency to learn standard algorithms by rote and to apply them "head-on" can do the student a disservice. Thus, the student who, after studying the basic course decides to do the "college" part of the exam as well, can turn out to be psychologically unprepared for such work.

The "difficult part" of the exam contains many problems pertaining to material that is shared by the basic and advanced courses. Their formulations are thus understandable to all students. However, without a developed capacity for mathematical thought, without the skills associated with advanced mathematical activity, it is unrealistic for students to hope to solve them. To some extent, it may be said that students who have completed the basic course, but who by the time they graduate from high school have decided for one or another reason to go on to colleges that require applicants to take a USE in mathematics, are thus given a chance to display their giftedness.

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5
Elements of Analysis in Russian Schools

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1 Introduction

In this chapter, the term "elements of analysis" will be used to refer not only to such topics as limits, derivatives, and integrals, which have traditionally belonged to the "course in higher mathematics" in Russia, but also to a much broader range of topics connected with the concept of functions.

The concept of functions is introduced explicitly in seventh or eighth grade (depending on the program and textbook). In basic school (up to ninth grade), students study in detail such crucial functions as the linear function, the quadratic function, and inverse proportionality $\left(y = \frac{k}{x}\right)$. Using the example of these functions, the general characteristics of functions are introduced and studied; these include domain and range, monotonicity, and whether the function is even or odd. Naturally, the whole investigation is conducted using elementary means. Along with functions defined by formulas, students consider functions given in the form of a graph or obtained by analyzing various real-world processes. The basic school course in algebra may be said to be grouped around three basic concepts: transformations, equations and inequalities, and functions.

In grades 10 and 11 in secondary school, the continuation of the course in algebra bears the official title "Algebra and Elementary

Calculus." Two developments take place here: the class of elementary functions that students investigate is expanded, as is the range of mathematical methods used to investigate functions.

Along with the functions studied earlier, students are now introduced to power functions (with an examination of fractional and negative exponents), exponential functions, logarithmic functions, and trigonometric functions. Functions are investigated using the methods of calculus.

Among the issues that have traditionally been part of the apparatus of calculus, the topic studied in the greatest detail is "Derivatives." The content here is sufficiently traditional: the derivatives of basic elementary functions, the rules of differentiation, and using derivatives to investigate functions and solve problems involving maxima and minima. If the topic "Derivatives" is studied by all upperclassmen, then matters become more complicated with the topic "Integrals." Only the study of the concept of "antiderivatives" is currently declared to be fully mandatory, along with the corresponding computation of antiderivatives, with more difficult problems (such as problems that involve computing areas) forming an optional part of the program — more precisely, they are currently not included in the so-called requirements for graduation, which means that such problems will not appear on exams.

In general, the fruitfulness of studying the elements of calculus in school is sometimes contested. Critics mainly point to the difficulty of this material. Some time ago, education minister Andrey Fursenko even saw fit to remark that studying the methods of higher mathematics kills students' creativity (http://www.rosbalt.ru/2009/02/11/617365.html). However, these sections have long ago and firmly become established as parts of the school curriculum and school practice, and the author of this chapter likewise unequivocally belongs to the number of their supporters, believing that the exposition of the elements of higher mathematics contributes greatly to students' intellectual development.

In this chapter, we will briefly present the history of the teaching of higher mathematics in Russian (Soviet) schools and concentrate on analyzing the main textbooks and programs, examining that component in them which pertains to elements of analysis. We should also note that in this chapter we will be focusing on so-called "ordinary" schools. The programs of schools with an advanced course in mathematics have traditionally devoted far greater attention to calculus, but this topic is addressed in other chapters of this volume.

2 Elements of Analysis in Normative Documents

The Standards currently in effect (Standards, 2009) address two sections connected with our topic: "Functions" and "Elementary Calculus." These sections are included in the basic minimum content of the basic curricula (material that must be studied but is not part of the graduation requirements is indicated in italics).

Functions

Functions. The domain and the range. The graph of a function. Constructing the graphs of functions defined in various ways. Properties of functions: monotonicity, evenness and oddness, periodicity, boundedness. Intervals of increase and decrease, global maxima and minima, local maxima and minima. Graphic interpretations. Examples of functional dependencies in real-world processes and phenomena.

The inverse function. *The domain and the range of the inverse function*. The graph of the inverse function.

Power functions with natural exponents, their properties and graphs.

Vertical and horizontal asymptotes of graphs. Graphs of functions defined by formulas $y = \frac{ax+b}{cx+d}$.

Trigonometric functions, their properties and graphs; periodicity, fundamental periods.

Exponential functions, their properties and graphs.

Logarithmic functions, their properties and graphs.

Transformations of graphs: parallel translation, symmetry with respect to the coordinate axes, and *symmetry with respect to the origin, symmetry with respect to the line* $y = x$, *stretching and shrinking along the coordinate axes.*

Elementary calculus

The concept of the limit of a sequence. The existence of a limit of a bounded monotonic sequence. The length of a circle and the area of a circle as limits of sequences. The infinitely decreasing geometric progression and its sum.

The concept of the continuity of a function.

The concept of the derivative of a function, the physical and geometric meaning of the derivative. The equation of a line tangent to the graph of a function. Derivatives of sums, differences, products, quotients. Derivatives of basic elementary functions. Using derivatives to investigate functions and construct graphs. *The derivative of the inverse function and of a composite of a given function with a linear function.*

The concept of a definite integral as the area of a curvilinear trapezoid. The antiderivative. The Newton–Leibniz formula.

Examples of using derivatives to find optimal solutions to applied problems, including socioeconomic problems. Finding the velocity of a process defined by a formula or graph. Examples of using integrals in physics and geometry. The second derivative and its physical meaning.

The Standards also stipulate that students must develop the following abilities in studying these sections:

Functions and graphs

Students must be able to:

- Define the value of a function based on the value of its argument for various ways in which a function is defined;
- Construct graphs of studied functions;
- Describe the behavior and properties of functions based on their graphs and *in elementary cases based on their formulas*, and find the greatest and least values of a function based on its graph;
- Solve equations, and elementary systems of equations, using *the properties of functions* and their graphs;

Use acquired knowledge and abilities in practical activities and everyday life to:

- Describe various dependencies using functions, represent them graphically, and interpret graphs.

Elementary calculus

Students must be able to:

- Compute the derivatives *and antiderivatives* of elementary functions, using reference materials;

- Investigate functions for monotonicity in elementary cases, find the greatest and least values of functions, and construct graphs of polynomials *and elementary rational functions* by using the apparatus of calculus;
- *Compute areas by using antiderivatives in elementary cases*;

Use acquired knowledge and abilities in practical activities and everyday life to:

- Solve applied problems, including socioeconomic and physical problems, that involve finding the greatest and least values, and determining velocity and acceleration.

It should be pointed out that the study of the concepts of calculus does not take place only in the courses "Algebra" and "Algebra and Elementary Calculus." A considerable amount also is covered in the course in geometry: such topics as "the length of a circle and the area of a circle" and "the volumes of objects and the areas of their surfaces" — the central topic of the 11th-grade course in geometry — require the use of the methods of calculus.

3 The History of Higher Mathematics Education in Russian (Soviet) Schools

The study of the concepts of analysis in Russian schools has a sufficiently long history. Below, following mainly the work of Savvina (2003), we briefly characterize its main periods.

3.1 *The Second Third of the 18th Century to 1845*

This period may be said to be characterized by the spontaneous introduction of elements of higher mathematics into teaching. It was a period of great freedom and variety in the curricula of educational institutions; the content of education was defined mainly by the textbooks in use, as well as by the tastes of individual teachers and their preparation. During this time, the first Russian language textbooks in higher mathematics were written. For example, in 1814, *Elementary Foundations of Pure Mathematics*, a textbook by Nicolas Fuss (student and relative of the great mathematician Leonhard Euler, who lived and worked in St. Petersburg), was recommended for use in instruction in gymnasia; Part III of this textbook contains an introduction to differential and integral calculus. However, the elements of higher mathematics did not become a standard part of the gymnasium curriculum. Calculus was introduced at the very end of the extensive gymnasium course in mathematics, and thus there was usually not enough time for it.

A more modern approach to mathematics education could be found in *real schools*, and particularly in secondary military academies. Thus, for example, already by the beginning of the 19th century, the course of instruction for the First and Second Cadet Corps contained differential and integral calculus.

3.2 *1846–1906*

In 1845, Minister of Public Education Sergey Uvarov published an "Official Proposal to Limit Mathematics Education in Gymnasia." Appended to this proposal was the first general mathematics curriculum for all Russian gymnasia, written by F. I. Busse. All applied topics in mathematics had been diligently expunged from this curriculum, including analytic and perspective geometry and the calculus of infinitesimals. The idea of a classical education crystallized (i.e. a form of education that emphasized the teaching of the ancient languages — Latin and Greek — on a considerably greater scale than the teaching of mathematics and the natural sciences). Looking ahead, we should say that up until the Revolution of 1917, higher mathematics was not taught in classical gymnasia.

3.3 *1907–1917*

The late 19th and early 20th centuries witnessed a broad discussion concerning the fruitfulness of including the ideas of higher mathematics in the school curriculum. Particular attention was devoted to this problem during the First and the Second All-Russian Congress of Mathematics Teachers, in 1911–1914. One of the items in the resolution passed by the First Congress concerns directly our topic:

> The Congress recognizes that the time has come to omit certain topics of secondary significance from the secondary school course in

> mathematics, to strongly emphasize and structure the course around the idea of functional dependency, and also — in the aims of bringing secondary school education closer to the demands of modern science and life — to acquaint students with the elementary and most accessible *ideas* of analytic geometry and calculus. (Publications, 1913; p. 571)

In the course of the discussion, educators determined a basic minimum of information about calculus that had to be included in the school course: the definition of the first and the second derivative of a function with one variable, the differentiation of polynomials and trigonometric functions, finding the derivatives of composite functions, geometric applications of derivatives, the concept of a definite and an indefinite integral, and integrating polynomials. About 30 textbooks on higher mathematics for secondary educational institutions came out at this time; among the authors of these textbooks was the outstanding mathematics educator A. P. Kiselev.

Citing arguments in support of the desirability of bringing the elements of higher mathematics into the schools, the mathematicians and mathematics educators of this time mentioned the fact that these topics presented scientific interest, developed students' thinking, prepared them for the college course in mathematics, had useful practical applications, and integrated Russian education into the world system of education.

Partly as a result of this discussion, the elements of higher mathematics during the first 15 years of the 20th century were included in the programs of various types of educational institutions: in 1907, in the curriculum of real schools; in 1911, in the cadets' corps curriculum; in 1914, in the curriculum of commercial schools. The classical gymnasium alone stalwartly resisted any attempts at modernization.

3.4 *1918–1933*

Following the October Revolution, which broke with the past in all spheres of life, schools also changed radically. The goal was now a new, mass education in "unified labor schools." The old system of education was criticized as feudal, not class-conscious, and alienated from life.

The goal of education changed as well: the acquisition of knowledge was no longer seen as an objective; rather, schools had to prepare students for life, for productive labor. The system of classes, lessons, and subjects in many respects lost its place in the schools. "Complex education," which was not divided into subjects, was established and promoted.

All of this led to the disappearance of a unified system of education and to a general fall in the level of knowledge, including mathematical knowledge. Curricula were regarded only as recommendations. Although in these curricula attempts were made to preserve the elements of higher mathematics in schools, they were not successful in practice.

3.5 *1934–1964*

The 1930s were the time of Stalin's counterreforms. Effectively, the government set itself the task of recreating the old gymnasium, naturally within a different ideological framework. The 1930s–1950s were the period of greatest stability in Soviet schools. Higher mathematics was not studied in school; however, school curricula included a preliminary preparation for the future study of calculus — in particular, students were introduced to the function-oriented approach and to the concept of the limit of a sequence.

During the 1930s–1950s, the possibility of including elements of calculus in the school course in mathematics was again a subject of discussion. In addition to arguments that had been heard in the discussion at the beginning of the 20th century, calculus supporters now pointed out that the study of differential and integral calculus facilitated the formation of a dialectical worldview. During the 1950s, another argument appeared: calculus was seen as a means for the "polytechnization" of education.

3.6 *1965–1976*

The middle of the 1960s marked the beginning of a radical transformation of Soviet mathematics education within the context of the well-known Kolmogorov reforms. The elements of calculus began to be widely promoted in schools, appearing in the programs of elective courses and mathematics circles. Gradually, this material was introduced into mass secondary schools as well.

Both differential and integral calculus, as well as preliminary material that introduced students to thinking in terms of functions, were broadly represented in the new plan for the secondary school mathematics curriculum (Kolmogorov *et al.*, 1967). In seventh grade (schools at the time had 10 grades, seventh grade corresponding to today's eighth grade), error estimation was studied in detail. Formulas expressing the errors of a sum, product, and quotient in terms of the errors of the components of these operations are in essence formulas of differential calculus. In ninth grade, students were introduced to the concept of the limit in the language of "$\varepsilon - \delta$." The distribution of class hours by topic in the course for grades 9 and 10 was as follows (Kolmogorov *et al.*, 1967):

Grade 9

1. Infinite sequences and limits (15 hours)
2. Continuous functions, the limit of a function, derivatives (45 hours)
3. Trigonometric functions, their graphs and derivatives (30 hours)

Grade 10

1. Derivatives of exponential and logarithmic functions (8 hours)
2. Integrals (12 hours)
3. Trigonometric functions (*continued*) (40 hours)

As can be seen, this program, which was largely accepted and implemented in school practice, accorded a central role to calculus in the last two grades of school. Issues regarding the content and order of exposition of the material were resolved basing on the results of practical teaching.

3.7 *1977 to the End of the 1980s*

During this period, the body of knowledge concerning higher mathematics conveyed in school stabilized. A stable textbook was introduced: *Algebra and Elementary Calculus*, edited by Andrey Kolmogorov

(Kolmogorov *et al.*, 1977). This textbook will be described in greater detail below (it has remained in use, albeit in a revised form, to this day). During a counterreform in the early 1980s, the elements of calculus, while remaining in the school curriculum, underwent reductions, often illogical and arbitrary ones. In particular, the concept of the limit was removed from the curriculum, although derivatives and integrals were retained — now deprived of a foundation.

3.8 *Early 1990s to the Present*

This has been a period of searching for and implementing different approaches to teaching the elements of calculus. A number of different textbooks have appeared as alternatives to Kolmogorov's textbook, embodying different approaches to structuring the course in "Algebra and Elementary Calculus." The first of these that we should mention is the repeatedly reissued series of textbooks by Sh. A. Alimov *et al.* (such as 1991a, 1991b, 1991c, 1992). We should also mention certain textbooks by other teams of contributors: in basic school, these include Dorofeev, Sharygin *et al.* (2007a, 2007b), Dorofeev, Suvorova *et al.* (2005, 2009a, 2009b), Makarychev *et al.* (2009a, 2009b, 2009c), and Mordkovich (2001a, 2001b, 2001c); in grades 10–11, Bashmakov (1991), Dorofeev, Kuznetsova *et al.* (2003, 2008), Dorofeev, Sedova *et al.* (2007, 2010), and Mordkovich and Smirnova (2009a, 2009b).

4 Introduction to Analysis: Functions in Basic School

Let us now examine the presentation of the topic of functions in grades 7–9 in basic school. The scope of this chapter does not permit us to provide a systematic description of the way in which this material is presented in all textbooks that are currently used; moreover, the situation is not stable. Not so long ago, for example, all kinds of power functions were still studied in considerable detail in ninth grade; today, a part of this material has been moved to tenth grade, and a part of it has been eliminated altogether. On the other hand, the most recent editions of textbooks do not always accurately reflect what actually goes on in schools. For example, although in the textbooks and curricula trigonometric functions have also been moved to tenth grade, many teachers — according to our observations — continue to begin the presentation of this topic in ninth grade, as before, believing that otherwise the course for grades 10–11 would become overloaded (recall that in Russia a class of students will often have the same teacher in grade 9 and grades 10–11). At present, such actions on the part of teachers are usually not interfered with by school administrators.

The amount of material pertaining to the theory of functions that is officially studied in grades 7–9 is not so great: linear functions, quadratic functions, functions of the type $y = \frac{k}{x}$, as well as certain general properties of functions. Below, we describe certain basic features of the way in which this material is studied, confining our examples to two series of textbooks.

As has already been said, three principal themes may be identified on the whole in the basic school course in algebra: transformations, equations and inequalities, and functions. Equations and functions are closely connected; equations are solved using the properties of functions, so it may be said that functions are primary and equations are secondary. However, the more traditional point of view requires that students begin with equations. This is the point of view adopted by the textbook of Alimov *et al.*

The concept of a function is introduced during the second semester of seventh grade, mainly on the example of linear functions. The presentation begins with a problem about motion that motivates what follows [we will give references to Alimov *et al.* (1991a), which was one of the first editions of this text; despite numerous subsequent editions, no fundamental changes in these sections have been made]:

> A train is moving from Moscow to St. Petersburg at a speed of 120 km/h. What distance will the train travel in t hours? (p. 124)

The answer is given in the form of a formula: $s = 120t$. Subsequently, the concept a *variable* is introduced: this is a "quantity that changes." In the given problem, the variables are time t and distance s; t is called an *independent variable* and s is called a *dependent variable* or *function*. The dependency of the variable s on the variable t is called a *functional dependency.*

In this way, no explicit definition of functions is given at this stage. The term "function" is not taken to mean dependency between two variables: from the viewpoint of the authors of the textbook, this is synonymous with the expression "dependent variable." Subsequently, three methods of defining functions are discussed: a function may be defined by a formula, table, or graph. In connection with the examination of the third method, the authors provide the definition of a graph:

> The graph of a function is defined as the set of all points in the coordinate plane whose x coordinates are equal to the values of the independent variable, and whose y coordinates are equal to the corresponding values of the function.

The problems given in this section are aimed at developing the following basic skills: finding the value of a function for a given value of x, finding the values of x for which the function assumes a given value of y, and finding several values of x for which the function is positive (negative). The functions in the problems are defined both by formulas and by graphs. The fact that y has a single value for any x, while x does not have a single value for any y, is not discussed or even mentioned.

Then, the textbook examines the linear function $y = kx + b$ and its graph. The fact that the graph of this function is a straight line is accepted without proofs; the students are simply told that it can be shown that it is a straight line (thus, at least the textbook expresses the thought that this is something which must be shown). For linear functions, the same typical problems that we mentioned earlier are solved; to them is added the problem of "constructing a graph." Below, we reproduce review problems pertaining to this material, which appear at the end of the section. Such problem sets (under the heading "Test yourself!") conclude each section in the textbooks by Alimov *et al.* They enable the students themselves (as well as their teachers) to test how well they have learned the basic "typical" skills:

- Given the function $y = 5x - 1$, find $y(0.2)$ and the value of x for which the value of this function is equal to 89. Does the point $A(-11, 54)$ belong to the graph of this function?
- Construct the graph of the following function: $y = 2x$; $y = x - 2$; $y = 3$; $y = 3 - 4x$. (Alimov *et al.*, 1991a, p. 145)

In the next chapter, "Systems of Linear Equations," the textbook examines three methods for solving such systems. Two of them are purely algebraic (substitution and algebraic addition), while the third method is graphic and uses the concept of a linear function and its graph. The graphic method is used not only for solving linear systems, but also for investigating (using geometric considerations) whether systems of two linear equations have solutions. The three possible ways in which two straight lines may be positioned with respect to one another in the plane — they can intersect, be parallel, or coincide — correspond to three kinds of solution sets for systems of two linear equations: one solution, no solutions, and an infinite number of solutions.

By comparison, the textbooks of Dorofeev, Suvorova *et al.* (2005, 2009a) are structured somewhat differently. Graphs appear before functions. Students are introduced to the concept of the coordinates of a point, and subsequently they are asked to construct various sets of points in the coordinate plane. In particular, it is brought to their notice that certain points in the coordinate plane lie on the same straight line; the conclusion is then drawn that the equation which the coordinates of the points satisfy may be said to define the straight line.

The problems offered in this textbook are somewhat more geometric than the problems in the textbook discussed above. But the text is by no means always aimed at eliciting from the students the confident demonstration of some acquired skill; more precisely, a part of the material is given not in order to develop any skill but to acquaint the students with the subject and broaden their horizons. Thus, for example, as early as seventh grade, students are introduced to the graph of the relations $y = x^2$ and even $y = x^3$ (the methodology is the same as the one used with the linear function: let us construct a table and connect the points; the word "function" is not used — the authors talk about relations between coordinates). This does not mean, however, that the teacher must teach the students to construct the graphs of quadratic functions; the point, rather, is that the students must understand from the beginning that graphs are not necessarily straight lines. We should note, in connection with this, that the textbooks by Dorofeev, Suvorova *et al.* (2005) devote considerable attention to the graphs that surround us, which may be curves of extreme intricacy. The students are asked to use a temperature graph, for example, to indicate the time when the temperature was equal to various given magnitudes, when it was highest, and when it was increasing or decreasing.

The concept of a function is introduced in the textbooks of this series in eighth grade (Dorofeev, Suvorova *et al.*, 2005). Here, too, variable values are mentioned, the domain of a function is defined, the graph of a function is discussed, and the vertical line test is effectively presented. Students are introduced to linear functions (now with the use of the term "function") and functions of the form $y = \frac{k}{x}$. In discussing graphs, the authors mention increasing and decreasing functions, taking the former to mean that the graph "in moving from the left to the right always goes up." Consequently, the students are given problems that require them to indicate, based on a graph, whether one or another function is increasing or decreasing.

In the textbook of Alimov *et al.*, quadratic functions are studied in eighth grade. By this time, quadratic equations and systems of quadratic equations, problems that can be reduced to quadratic equations, and the like have already been studied in detail by the students. Now, the same kinds of problems are solved in connection with quadratic functions as were solved earlier in connection with linear functions: find the value of the function for a given x and the values of x for which the function assumes a given value. The second of these kinds of problems naturally requires students to solve a quadratic equation; however, this textbook makes no general statement about the fact that solving such a problem is the same as solving an equation.

A central position in this chapter of this textbook is occupied by the construction of graphs of quadratic functions. The graph of the function $y = x^2$ is constructed "point by point"; students are then asked to connect the points with a "smooth" curve and told that the resulting curve is called a "parabola." The elementary properties of the function $y = x^2$ are enumerated, including its nonnegativity, the symmetry of its graph with respect to the coordinate axis, and the fact that it is increasing for $x \geq 0$ and decreasing for $x \leq 0$. Then the function $y = ax^2$ is examined. The textbook points out that its graph can be obtained from the graph of $y = x^2$ by means of expansion or contraction, and for negative a also by means of reflection. The construction of the graph of a quadratic function in the general case is connected with the construction of the graph of the function $y = ax^2$ and grounded theoretically (this is done by completing the perfect square out of the expression $ax^2 + bx + c$, which makes it possible to find the coordinates of the vertex of the parabola, and then by examining translations along the coordinate axes). Subsequently, however, in constructing graphs, the textbook usually confines itself to using the formula for the x coordinate of the vertex $x_0 = -\frac{b}{2a}$, although teachers usually require students to indicate several other key points as well — first and foremost, the points where the graph and the coordinate axes intersect.

While constructing the graph of $y = x^2$, the textbook introduces the terms "increasing" and "decreasing": students are told that a function is increasing when a greater value of y corresponds to a greater value of x; but, again, statements about the concepts of increasing and decreasing are made with reference to a graph and with the use of examples.

Among the typical problems given to students in association with this topic are not only problems that involve constructing graphs of quadratic functions, but also problems that involve investigating the properties of quadratic functions with the help of these graphs. In particular, students are asked to find the least value of a given function; to find the values of x for which the value of the function is equal to a given number (for example, 3); to find the values of x for which a function assumes positive (negative) values; to determine the intervals where the given function increases (decreases); and to find the coordinates of a vertex.

As in the case of the linear function, the concept of the quadratic function is immediately applied to algebraic material. Thus, in the section on "Quadratic Inequalities," three methods for solving such inequalities are described: by factoring the quadratic trinomial and solving a system of inequalities; by using the graph of the quadratic function; and by using the so-called interval method.

In the textbooks of Dorofeev, Suvorova *et al.* (2009b), quadratic functions are studied in ninth grade. Here, again, there are more geometric formulations, more attention to symmetry, translation, and so on.

In the textbooks of Alimov *et al.*, the general properties of functions were originally systematically studied in ninth grade in connection with the topic "Power Functions." As has already been noted, over the past 10–15 years all kinds of possible abridgments have gradually crept into the text here (connected first and foremost with an abridgment — or, more precisely, termination — of the study of powers with rational exponents). Nonetheless, today, teachers in ninth grade still usually not only speak about functions of the form $y = \frac{k}{x}$, but also provide certain general definitions (the domain of a function).

A typical lesson on the given topic could have (and even, in part, still may have) the following form [Dobrova, Lungardt *et al.* (1986) or later editions]:

> At the beginning of the lesson, the class again goes over solving simple linear inequalities. The teacher then examines the expressions
>
> $$x^2 - 2x + 3, \frac{1}{x-2}, \sqrt{x}$$
>
> and calls the students' attention to the fact that the last two of these expressions are determined not for all values of x. After this, a definition of the concept being studied is formulated.
>
> The set of all values that the argument of a function can assume is called *the domain of the function*.
>
> Then, the following problems from the textbook are examined: find the domain of the following functions: (1) $y = 2x^2 + 3x + 5$; (2) $y(x) = \sqrt{x-1}$; (3) $y(x) = \frac{1}{x+2}$; (4) $y(x) = \sqrt[4]{\frac{x+2}{x-2}}$.
>
> The solution to problem (4) is more difficult than the solutions to the others (note that, according to the curriculum in use today,

> fourth roots are not studied in the nine-year program) and offers an opportunity once again to go over the topic of solving inequalities by using the interval method, which is sufficiently difficult for many students. The teacher may focus the students' attention on the fact that they are familiar with two "dangerous" operations, which are the reasons for the bounds placed on a function's domain: division (it is impossible to divide by 0) and extraction of even roots (it is impossible to extract an even root of a negative number). Then the students solve problems that involve finding the domains of various functions. As homework, the students may be given problems similar to the ones solved in class, such as "Find the domains of the functions $y(x) = \frac{2x}{x^2-2x-3}$; $y(x) = \sqrt{3x^2 - 2x + 5}$"; and problems that involve repeating what has been covered, such as "A function is defined by the formula $y(x) = \frac{x+5}{x-1}$; find $y(0)$, $y(-2)$; find the value of x if $y(x) = -3$, $y(x) = 13$."

This lesson can be given in 10th grade as well, but we cite it here because it is very representative of one of the possible directions which the study of functions may take. Along with introducing and developing a new theoretical concept, considerable attention is devoted to going over topics that involve solving linear and quadratic inequalities and developing the corresponding skills. The *algebraic* element here probably outweighs the *analytic* element. As we have seen, other textbooks attempt to make the course more qualitative, geometric, and visual, and less technical and formula-laden. A strong point of both sets of textbooks, in our view, lies in their striving to underscore connections with other sections of the course.

It was noted above that no rigorous proofs are given for most of the assertions that are made concerning functions. Still, some attempt is usually made to give examples or to provide some kind of *plausible argument* in support of the assertions being made, and this is another positive point [although the dynamic here is a complicated one: one might recall that 40–50 years ago school textbooks for the eight-year schools of the time contained, for example, a practically precise proof of the fact that the graph of the function $y = kx$ is a straight line (Barsukov, 1966)].

5 Algebra and Elementary Calculus: Functions in Grades 10–11

The course "Algebra and Elementary Calculus" for grades 10–11 roughly corresponds, for example, to the American courses "Algebra II," "Precalculus," and "Calculus," but with a number of differences. Below, we will systematically discuss the study of functions without the use of differential calculus, following the textbooks of Alimov *et al.*,[1] then touch on teaching the elements of calculus, and in conclusion talk about certain textbooks that have appeared relatively recently.

As has already been said, much of what used to be studied in grade 9, and then in grade 10 on a higher level, is now studied only in grade 10. The textbook of Alimov *et al.* (2001) contains chapters on "Real Numbers," "Power Functions," "Exponential Functions," "Logarithmic Functions," "Trigonometric Formulas," "Trigonometric Equations," "Trigonometric Functions," and three more chapters, devoted to calculus. Thus, the theme of functions may be said to be the central theme of the course for grades 10–11. At the same time, for example, of the five sections in the chapter on "Power Functions," three are devoted to equations and inequalities — "Equivalent Equations and Inequalities," "Irrational Equations," and "Irrational Inequalities" (for optional study). Only two sections are devoted to functions themselves — "Power Functions and Their Graphs" and "Functions That Are Inverses of Each Other."

Almost without commentary, the textbook lists the properties of various power functions (domain, range, evenness and oddness, increasing and decreasing), providing a "representative" graph for each case (in discussing this topic in class, the teacher will most likely begin precisely with a concrete graph, indicating several points and then drawing a conclusion about the behavior of the function). The

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[1] In the latest editions of this textbook, its lead author has changed: the head of the team of contributors is now Yu. M. Kolyagin. The textbooks of Kolyagin *et al.* (2007a, 2007b) are very similar to that of Alimov *et al.* (2001) in terms of their material and presentation; therefore, here and below we will confine our discussion to the latter.

problems solved in studying this topic revolve around the properties of various power functions, such as:

- Schematically depict the graph of the following function and indicate its domain and range: $y = x^6$, $y = x^{\frac{1}{2}}$, $y = x^{-3}$;
- Using the properties of the power function, compare $0.2^{0.3}$ with 1;
- Find the intervals in which the graph of the function $y = x^{1-\pi}$ lies above (below) the graph of the function $y = x$. (Alimov *et al.*, 2001, pp. 44–45)

Material pertaining to the power function is used to introduce the concept of an inverse function. The main examples here are, of course, the functions $y = x^3$ and $y = x^2$, $x \geq 0$. The presentation is conducted in a sufficiently "scientific" manner: the concept of an invertible function is explicitly introduced (in essence, injectivity), and the theorem that monotonic functions are invertible is formulated and proven. Also proven is the theorem that the graphs of a function and its inverse are symmetric (it is another matter that the teacher will by no means always present this proof in class, let alone ask the students to reproduce it).

A proper exposition of the topic "Exponential Functions" requires the concept of a power with an arbitrary real exponent. This in turn forces the authors to introduce the concept of a limit (which, however, is also used elsewhere for defining and finding the sum of an infinitely decreasing geometric progression). All of this is done in the textbook's first chapter, "Real Numbers." The concept of a limit is introduced using examples of progressive approximations of irrational numbers; in the process, students are acquainted with the necessary notation (lim) and some terminology. The presentation is very concise and the students are given practically no problems involving the independent finding of limits, so there is little reason to expect that this concept will be grasped with any depth. In the same chapter, "Real Numbers," the authors of the textbook define a power with an irrational exponent a^x as the limit of the sequence a^{x_n}, where x_n is the nth decimal approximation of x. It is explicitly stated that the existence of this limit — and the fact that for a power defined in this way, all known properties of powers hold — "is demonstrated in the course in higher mathematics." At the same time, the textbook does contain a number of proofs of properties of powers that rely on properties already formulated; for example, it is proven that for $a > 1$ and $x_1 < x_2$, the inequality $a^{x_1} < a^{x_2}$ holds (based on the fact that for $a > 1$ and positive t, the inequality $a^t > 1$ holds).

The chapter on "Exponential Functions" then derives the properties of exponential functions: the fact that their domain is the set of all real numbers, that their range is the set of all positive numbers (in this connection, the authors make use of another fact that "will be proven in the course in higher mathematics": the fact that the equation $y = a^x$ has a solution for any $b > 0$); and the fact that the function is increasing over the entire set of real numbers for $a > 1$ and decreasing over this set for $0 < a < 1$. Graphs are constructed in connection with all of these considerations.

Typical problems that are mandatory for all students include the following:

- Schematically depict the graph of the function $y = 0.4^x$;
- Find the coordinates of the points of intersection of the graphs of the functions $y = 2^x$ and $y = 8$;
- Determine whether the function $y = 0.7^{-3x}$ is increasing or decreasing. (Alimov *et al.*, 2001, p. 74)

Among the difficult problems is, for example, the following:

- Find the greatest and least values of the function $y = 2^{|x|}$ on the segment $[-1, 1]$. (Alimov *et al.*, 2001, p. 75)

Subsequently, in the exposition of the topics "Exponential Equations" and "Exponential Inequalities," the properties of powers and exponential functions are used extensively. Thus, the solution of the equation $a^x = a^b$ makes use of a property of a power that is effectively equivalent to the injectivity (invertibility) of the function $y = a^x$; the solution of inequalities such as $a^x > a^b$ or $a^x < a^b$ relies on the fact that the exponential function is increasing or decreasing.

The chapter on "Logarithmic Functions" is structured, in a typical manner for this textbook, in accordance with the same schema. First, students are introduced to the algebraic concept of the logarithm of a positive number b with the base a as the unique solution of the equation $a^x = b$ (the uniqueness of this solution was derived earlier from the monotonicity of the exponential function). Properties of logarithms are proven, after which the logarithmic function $y = \log_a x$ is defined. The properties of this function — its domain, range, monotonicity, and signs — are derived from the algebraic properties of logarithms. The injectivity of the logarithmic function is proven on the basis of its monotonicity: if $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$ $(a > 0,\ a \neq 1,\ x_1 > 0,\ x_2 > 0)$. This fact is used as a foundation for solving logarithmic equations. A graph of the logarithmic function is constructed on the basis of properties that have been proven. Lastly, the assertion is made that the exponential and logarithmic functions are inverse functions. Typical problems pertaining to the topic "Logarithmic Functions" include the following:

- Construct the graph of the function $y = \log_2 x$; $y = \log_{\frac{1}{2}} x$;
- Find the domain of the function

$$y = \log_4(x - 1);\ y = \log_3(x^2 + 2x).$$

Problems that are labeled as more difficult include the following:

- Prove that the function $y = \log_2(x^2 - 1)$ is increasing over the interval $x > 1$;
- Find the domain of the function $y = \log_\pi(2^x - 2)$;
- Construct the graph, and find the domain and range of the following function: $y = 1 + \log_3(x - 1)$. (Alimov *et al.*, 2001, p. 102)

Among the trigonometric chapters, only the chapter "Trigonometric Functions" is directly related to our subject; however, the chapters devoted to formulas and equations provide the necessary preparatory material for defining and investigating trigonometric functions. The sine, cosine, and tangent of an arbitrary angle α, measured in radians, are defined by means of the rotation of a point $P(1,\ 0)$ on the unit circle through α radians (the sine, cosine, and tangent of an acute angle have already been defined in the course in geometry). This construction is used to establish a correspondence between the points of the real number line and the points of the unit circle. The sine and cosine of an angle are defined as the y coordinate and x coordinate of a point obtained by means of a rotation of the point $P(1, 0)$ through angle α. We should emphasize that students are given definitions of the sine and cosine of an angle, not of a real number. Subsequently, however, they are informed that in the expressions $\sin\alpha$ and $\cos\alpha$, α can be regarded as a number. Effectively, the sine of a real number x is defined as the sine of an angle of x radians. Unfortunately, this important definition is not made explicitly. In the exposition that follows, the sines and cosines of angles and numbers appear indiscriminately mixed together. In solving trigonometric equations, students are introduced to the arcsine and arccosine of a number as the roots of the corresponding equations on certain intervals. In addition to equations, certain inequalities are solved; inequalities are solved with the help of the unit circle, effectively, from the definitions of sines, cosines, and tangents. Note that, here, the ordinary program includes sufficiently intricate equations [one example from a level mandatory for all students: $\sin x \cdot \sin 5x - \sin^2 x = 0$ (Alimov *et al.*, 2001, p. 194)].

The chapter "Trigonometric Functions" contains definitions of the functions $y = \sin x$ and $y = \cos x$: to each real number there corresponds a unique point on the circle; to the point there corresponds an angle; and to the angle, a sine and a cosine. The tangent function is defined as $\tan x = \frac{\sin x}{\cos x}$. Students solve problems that involve finding the domains and ranges of these functions.

The range of a function is determined by solving an equation with a parameter. Indeed, the number a falls within the range of the function $y = f(x)$ if and only if the equation $f(x) = a$ is solvable. The authors of the textbook use this argument not only to indicate the ranges of the basic functions $y = \sin x$ and $y = \cos x$, but also to solve a rather difficult problem about the range of the function $y = 3\sin x + 4\cos x$ (Alimov *et al.*, 2001, p. 199).

The textbook gives definitions of such general properties of functions as being even, odd, and periodic. Only following this are the graphs of the functions $y = \cos x$, $y = \sin x$, and $y = \tan x$ constructed and their properties enumerated. Inverse trigonometric functions are introduced as optional material. These functions are presented as inverse functions to trigonometric functions on corresponding intervals.

In this way, in five years of schooling (grades 7–11), students become acquainted with all of the basic elementary functions [as they are called, for example, in the classic calculus textbook of Fikhtengolts (2001)]. As we can see, for the authors of the textbook discussed here, functions are secondary compared with the corresponding equations. The well-developed apparatus of calculus, covered in 11th grade, is not used in the study of elementary functions: these two parts of the course are studied separately.

The textbook of Alimov *et al.* follows the classic Russian scheme: so-called elementary mathematics comes first (even if it is necessary to add to it just a little bit of the nonelementary — limits). As already stated above, in the "Stalinist" schools of the 1930s–1950s, derivatives were not studied. Their appearance effectively almost coincided with Kolmogorov's reforms (although the first textbooks in which the elements of calculus appeared were published before Kolmogorov's).

6 Elements of Differential and Integral Calculus

In discussing the study of calculus in Russian schools, it must be emphasized once more that this subject forms a part of the required course in mathematics for all students. Calculus must be studied by all students in the higher grades, not just some select group. It should also be borne in mind that this subject is studied today in 11th or even 10th grade, i.e. by 16–17-year-olds or even 15–16-year-olds. All of this heightens the tension between the wish to present the material at a high level of scientific seriousness and the requirement that it remain accessible to students.

From the point of view of science, calculus rests on a foundation created by Cauchy and Weierstrass in the 19th century, i.e. on the theory of real numbers and limits. As is well known, Cauchy's definition of the limit (in the language of "$\varepsilon - \delta$") makes it possible to prove the basic theorems of differential calculus rigorously. At the same time, this definition is extremely unintuitive and difficult to grasp not only for schoolchildren but also for certain college students.

When the elements of calculus were introduced into mass education during the time of the Kolmogorov reforms, someone may have believed that the difficulties connected with presenting the definition of limits were surmountable. Experience, however, forced educators to recognize that not all (or even almost all) students understand this definition. This led to the decision, as we have seen, virtually to strike the concept of the limit of a function off the school curriculum, while the relatively detailed study of derivatives *based on the concept of the limit*, as well as the considerably less detailed study of integrals, were retained. In this way, the edifice of school calculus must be erected without a foundation. Below, we will examine how various authors of textbooks handled this task.

6.1 *Andrey Kolmogorov's Textbook*

We first refer to the textbook by Kolmogorov *et al.* (1990), which was subsequently reissued without any changes that we would consider fundamental, and which was itself a version of a textbook by the same authors, Kolmogorov *et al.* (1977), revised in accordance with curriculum changes. The textbook discussed here has preserved to the greatest extent the ideas on which Kolmogorov's reforms were based. The distinctive feature of Kolmogorov's textbooks, in our view, is that they devote greater attention to explaining concepts than to developing students' command of techniques.

The chapter on "Derivatives and Their Applications" opens with a discussion on the concept of the "change of a function." The notations Δx and $\Delta f = \Delta y$ are introduced; it is emphasized that for a fixed value of x, the change Δf is a function of Δx. Examples are given of finding Δf as a function of Δx. Students practice solving problems of this type. Lastly, the textbook examines the geometric and physical meanings of the ratio $\frac{\Delta f}{\Delta x}$ as the slope of a secant and average velocity. In this way, the concept of change, which in other approaches is simply a tool used to define the derivative, acquires here an independent meaning.

The introduction of the derivative — the central concept of calculus — is preceded by a discussion on the concept of the tangent.

It is introduced visually: the tangent is said to be a straight line with which the graph of a function "practically converges." It is then argued that the slope of the tangent is a number to which the slope of the secant $k = \frac{\Delta y}{\Delta x}$ comes infinitesimally close. The discussion concludes with a definition of the derivative:

> The derivative of a function f at a point x_0 is defined as a number which the difference quotient $\frac{\Delta f}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ approaches when Δx approaches zero. (Kolmogorov *et al.*, 1990, p. 103)

Note that neither the word "limit" nor the sign for the limit is used. In the next paragraph, "The Concept of the Continuity of a Function and the Passage to the Limit," the definition of the limit does effectively appear: "the function f approaches the number L for x that approaches x_0 if the difference $f(x) - L$ is infinitesimally small, i.e. $|f(x) - L|$ becomes less than any fixed $h > 0$ as $|\Delta x|$ decreases" (Kolmogorov *et al.*, 1990, p. 106). The passage to the limit is used in two basic cases: in finding the derivative and in investigating the continuity of a function. This concept is also used to prove the continuity of the function $f(x) = \sqrt{x}$ from the definition.

The authors go on to provide rules for finding derivatives; using the formula for the derivatives of products and quotients, the formula $(x^n)' = nx^{n-1}$ for all integers n is proven by induction. The formula for the derivative of a composite function is discussed and proven. Students are asked to differentiate the following functions:

$$f(x) = (5x-2)^{13} - (4x+7)^{-6},\ f(x) = (x^3 - 2x^2 + 3)^{17} \text{(pp. 117–118).}$$

The formula for the derivative of the sine is derived using the limit $\left(\frac{\sin x}{x}\right) \to 1$ as $x \to 0$, which in turn is based on geometric considerations.

The textbook then discusses using continuity and the derivative. For continuous functions, the following property, obvious from visual considerations, is formulated without proof: "If the function f is continuous and does not become zero on an interval $(a;\, b)$, then its sign remains constant on this interval" (p. 122). Effectively, this is the intermediate value theorem, familiar from courses in calculus. This property is used as a foundation for the interval method and the method for finding approximate solutions to equations by progressively narrowing the segment at whose endpoints the function has different signs.

Geometric and mechanical applications of derivatives are examined. Among the geometric applications, for example, is the equation for the line tangent to the graph of a function, which in turn is used for approximate computations: for small Δx, the following approximate formula is defined: $f(x) \approx f(x_0) + f'(x_0)\Delta x$. This formula is used to find approximate values of power and trigonometric functions. Visual-geometric considerations underpin the so-called Lagrange formula (concerning the fact that when certain conditions hold on the segment $[a, b]$, there will be a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$).

The physical applications are quite varied and not reducible to the concept of instantaneous velocity. The textbook examines acceleration, linear density, and angular velocity. Of interest is the physical derivation (or illustration) of certain calculus theorems. The formula for the derivative of a sum is connected with the velocity addition law. Lagrange's theorem, mentioned above, is connected with the fact that at a certain moment in a motion, the instantaneous and average velocities must coincide.

The use of the derivative for the investigation of functions is the most important application of the derivative in the school curriculum. The presentation is quite similar to the college course in calculus, since it involves the formulation of Lagrange's theorem, which is fundamental to this topic. Lagrange's theorem is then used to prove sufficient conditions for a function to be increasing or decreasing; this is followed by a proof of a necessary condition for a function to have an extremum and sufficient conditions for a function to have an extremum (a change in the sign of the derivative). The final topic in the chapter "Derivatives and Their Applications" is the greatest and the least value of a function. A large number of applied problems are solved, providing the occasion to discuss mathematical modeling. An example of such a problem is:

> A square sheet of tin with side a must be used to make an open-top box by cutting out squares at the corners and bending the edges upward. What must be the length of the side of the base of the box in order for the box to have the maximal volume? (p. 152)

The next chapter of the textbook is called "Antiderivatives and Integrals." The problem that motivates the introduction of the concept of the antiderivative is taken from mechanics: "Given the acceleration of an object, find its velocity and coordinates at each moment in time." The subsequent presentation is sufficiently traditional. The antiderivative $F(x)$ of a function $f(x)$ is defined on an interval by the equality $F'(x) = f(x)$. The theorem that all antiderivatives of the function $f(x)$ on an interval have the form $F(x) + C$ is explicitly formulated and proven using Lagrange's theorem. A table of antiderivatives is obtained by means of an inversion of the table of derivatives. Three rules for finding antiderivatives are formulated and proven by differentiation: the sum of antiderivatives is the antiderivative of a sum; if F is the antiderivative of f, then kF is the antiderivative of kf; and if F is the antiderivative of f, $k \neq 0$, then $\frac{1}{k}F(kx + b)$ is the antiderivative of $f(kx + b)$. In this way, the formula for the substitution of a variable is introduced only in the linear case. Note that students are not introduced to the concept of an indefinite integral as the set of all antiderivatives or to the notation $\int f(x)dx$.

The concept of the integral is introduced in the textbook in an interesting way. First, the following theorem about the area of a curvilinear trapezoid is proven (the existence of this area is considered intuitively obvious and thus not discussed).

> Theorem. *If f is a function that is continuous and nonnegative on the interval* $[a; b]$, *and F is its antiderivative on this interval, then the area S of the corresponding curvilinear trapezoid is equal to the change in the antiderivative over the interval* $[a; b]$, *i.e.* $S = F(b) - F(a)$. (p. 180)

The theorem is proven using the definition of the derivative, while the change in area ΔS — the area of a "narrow strip" between two straight lines with x coordinates x and $x + \Delta x$ — is replaced with the area of the rectangle $f(c)\Delta x$, which is equal to it (the existence of such a rectangle is justified by citing the continuity of the function). Hence $\frac{\Delta S}{\Delta x} = f(c) \to f(x)$ for $\Delta x \to 0$ (here, the continuity of the function is used once again). In this way, the Newton–Leibniz formula is proven (using geometric language) even before the formal introduction of the concept of the integral.

The integral is then introduced as the limit of integral sums of a particular kind. The interval $[a, b]$ is divided into n equal parts, and the value of the function is taken at the left endpoint of each of the intervals thus formed. It is claimed that the sequence $S_n = \frac{b-a}{n}(f(x_0) + \cdots + f(x_{n-1}))$ approaches the area of a curvilinear trapezoid. Students are then informed that precisely this limit (which exists for any continuous function) is called the integral. Applications of integrals in geometry and physics are examined. To compute the volume of objects, the textbook introduces the formula $V = \int_a^b S(x)dx$, where $S(x)$ is the cross-section of an object with x-coordinate x, continuously dependent on x. Let us note, by the way, that in the course in geometry, the volumes of all studied objects, beginning with the pyramid, are usually computed using integrals (Atanasyan *et al.*, 2006). Among the physical problems solved using integrals is the problem of work done by a variable force, the problem of the force of the water pressure, and the problem of the centers of masses.

Finally, we should note that in contrast to the textbook by Alimov *et al.*, examined above, exponential and logarithmic functions are studied in this textbook after derivatives and integrals. The differentiation of the exponential function is initially carried out on the function $y = e^x$. The number e is introduced in the following way:

Examining the graphs of the functions $y = a^x$ for different a between 2 and 3, we notice that the slopes of the tangents to these functions at the point (0, 1) increase, passing through, as be might supposed from geometric considerations, the value 45° (whose tangent is equal to 1). The textbook concludes:

> It appears evident that as a increases from 2 to 3, we will find a value of a such that the slope will be…equal to 1. (p. 241)

After which the corresponding value of a is called the number e. In other words, e is defined as a number such that $\frac{e^{\Delta x}-1}{\Delta x} \to 1$ for $\Delta x \to 0$. From this equality, the formulas for the derivatives e^x and a^x, and also for the antiderivatives of these functions, are easily deduced. Then the derivative of the function $y = \log x$ is derived by differentiating the basic logarithmic identity $x = e^{\log x}$, and the derivative of the power function with an arbitrary real exponent is obtained as the derivative of a composite function. Subsequent study of the properties of elementary functions can be conducted using derivatives.

Let us note that the textbook of Kolmogorov *et al.* (1990) also touches on differential equations: it examines equations of exponential growth and decay, which lead to a function such as $f(x) = Ce^{kx}$, and the equation of harmonic oscillations, which leads to the function $f(x) = A\cos(\omega x + \phi)$.

In the opinion of the author of this chapter, the textbook of Kolmogorov *et al.* solved an extremely difficult methodological problem with considerable success: it presented elementary calculus in a way that is understandable and sufficiently rigorous. No doubt, there is little reason to suppose that references to the passage to the limit are always comprehensible to all students, but many topics are presented in a clear way and with great methodological and mathematical inventiveness. Very critical judgments of this textbook, however, have also been expressed (see also Abramov, 2010). This textbook has remained in print (with certain changes) to this day and plays a role in the educational process along with other textbooks.

6.2 *The Textbooks of Alimov et al. and Kolyagin et al.*

The textbook of Alimov *et al.* (2001),[2] which has been mentioned above, covers almost the same body of material in elementary calculus as the textbook of Kolmogorov *et al.* Therefore, we will focus mainly on the differences between their approaches.

The first chapter, devoted to calculus, is called "Derivatives and Their Geometric Meaning." Derivatives are introduced immediately following the examination of a problem about instantaneous velocity. The derivative is introduced as the limit of a difference quotient, with the use of the word "limit" and the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

[2] As has already been noted, the textbook of Kolyagin *et al.* (2007a, 2007b) is extremely similar to the textbook of Alimov *et al.* (2001), and thus we will confine our discussion to the latter.

The formulas $C' = 0$, $(kx + l)' = k$, $\left(x^2\right)' = 2x$, and $\left(x^3\right)' = 3x^2$ are proven from the definition, and it is taken for granted that, for example, $h \to 0$ implies that $h^2 \to 0, 3xh \to 0$, and $3x^2 + 3xh + h^2 \to 3x^2$ (p. 227).

Somewhat later, the authors announce that limits are not a part of the secondary school curriculum, and *for this reason* (our italics) certain proofs are not given or are not carried out rigorously (p. 228). The authors then go on to define limits anyway, in the language of "$\varepsilon - \delta$," and even offer a definition of continuity, explaining that continuity does not imply differentiability.

The textbook then examines several more examples of differentiation, after which it presents (without proof or discussion) the formula $\left(x^p\right)' = px^{p-1}$ for any real exponent. To some extent, by analogy with formulas that have already been formulated, the formula $\left((kx + b)^p\right)' = pk(kx + b)^{p-1}$ is given. The formulas for the derivative of a sum and for factoring out a constant are proven, but their proofs are labeled as optional (supplementary, more difficult material). The formulas for the derivative of a product and a quotient are not proven at all, although tested on an example. The derivative of the composite function is also presented without a proof.

The number e has already been introduced earlier, simply as a certain remarkable number. Now, without any discussion, it is announced that in courses in higher mathematics (i.e. in college), it is proven that $(e^x)' = e^x$, after which the derivative of the exponential function is defined in the general case. Similarly, the formula $(\log x)' = \frac{1}{x}$ is presented in finished form, after which the derivative of the logarithmic function with an arbitrary base is defined. For the derivative of a sine, a sketch of a proof is given and mention is made that it is possible to prove the equality $\lim\limits_{t \to 0} \frac{\sin t}{t} = 1$. The other trigonometric functions are presented simply as ready-made formulas.

It may be said that the order in which investigation of functions or "Antiderivatives and Integrals" are studied is practically the same as in the textbook of Kolmogorov *et al.* The difference, however, is that all of the necessary formulas, such as Lagrange's formula, are presented with an explicit clarification that their proofs appear in the course in higher mathematics, which — even if a geometric illustration is later given for an assertion that is made (as is the case with Lagrange's formula) — relegates it to the category of the optional. Note, however, that this textbook does present one more algorithm: an algorithm for testing functions for convexity (admittedly, in a section not required for general study).

In general, it may be argued that this textbook's strong point is its development of students' technical abilities, including the ability to differentiate, construct various graphs, and so on. In essence, the authors explicitly state that defining the difficult concepts of calculus is not their concern, and that their concern is to teach students to solve certain classes of problems that involve these concepts.

6.3 *M. I. Bashmakov's Textbook*

In contrast to virtually all other textbooks, the textbook of M. I. Bashmakov (1991 and other editions) has only one author: the well-known mathematician and methodologist M. I. Bashmakov. The first version of the textbook was intended for use in vocational schools, which in Soviet times also offered a complete secondary school course. It may be supposed that this partly explains the author's attention to physical and technological applications of mathematics. As the author emphasizes: "This textbook will teach you to use such mathematical instruments as functions and their graphs, derivatives and integrals, equations and inequalities"; "mathematical arguments and proofs...play the role of instructions and descriptions" (p. 3). The structure of the textbook is of interest. It is divided into six chapters. "Each chapter opens with an introductory conversation that leads up to the appearance of new basic concepts. At the end of each chapter is a concluding conversation, which includes information that is not required for study, but which may help the inquisitive person" (p. 4).

About the derivative, the textbook, following Newton, states that "the derivative is velocity" (p. 65). The author discusses how the concept of an (instantaneous) velocity is logically completely unobvious and requires the passage to the limit. In this way, the equality $v(t) = \lim\limits_{t_1 \to t} \frac{s(t_1)-s(t)}{t_1-t}$ is introduced (without a formalization of the concept of the limit). Then the derivative is introduced, following Leibniz, as the slope of a tangent; from these two examples, a new operation is derived — differentiation — and the ordinary definition of the derivative is given.

In investigating a function, the criteria for monotonicity and for the presence of extrema are presented using a mechanical interpretation of the derivative as velocity. In contrast to most other textbooks, considerable attention is devoted here to the concept of the differential as the principal part of the change of a function. Numerous physical applications are examined in accordance with the same schema: if the differential of one physical magnitude — such as, work — is proportional to the differential of another physical magnitude — such as displacement, $dA = F(x)dx$ — then force equals the derivative of work with respect to displacement, $F(x) = \frac{dA}{dx}$. The concluding conversation introduces the concept of linearization: a small change in one magnitude brings about a proportional change in another.

Trigonometric functions are introduced in connection with the description of periodic processes; in particular, the author examines uniform motion along a circle. Formulas for the derivatives of the sine and cosine are derived using the coordinates of the vector of the instantaneous velocity, which is perpendicular to the radius vector of a point on the circle. Using these formulas, the author finds approximate formulas for computing sines and cosines for small x: $\sin x \approx x$, $\cos x \approx 1 - \frac{x^2}{2}$.

The integral is *defined* as the area of a curvilinear trapezoid, after which the Newton–Leibniz formula related to its connection with antiderivatives is proven. It is then stated that the integral can be defined in four ways: as the area under the curve of a function, as the limit of sums, as the change in the antiderivative, and as a function of an interval.

In this way, Bashmakov's textbook introduces the material "on a physical level of rigor," maintaining this level throughout the text. In spirit, this textbook is closer to the calculus of Newton and Leibniz than the calculus of Cauchy and Weierstrass. At the same time, both the conceptual aspects and the techniques are laid out clearly and comprehensively. The language of the textbook is free and not "scientific-sounding" (for example, the chapter "Equations and Inequalities" bears an epigraph from George Orwell: "All animals are equal, but some animals are more equal than others"). In order for students to understand this textbook, however, it is desirable that they should have a decent knowledge of physics, which is not always the case.

6.4 *New Generation Textbooks*

The textbooks discussed above first appeared in the 1970s or 1980s. Below, we briefly describe certain textbooks that appeared and became popular significantly later.

6.4.1 *The textbook of A. G. Mordkovich and I. M. Smirnova*

The textbooks of Mordkovich and Smirnova (2009a, 2009b) conclude the series of textbooks by Mordkovich for grades 7–9. Their textbook is in many respects intended for independent work. "Each paragraph contains a detailed and comprehensive presentation of theoretical material, addressed directly to students" (Mordkovich and Smirnova, 2009a; p. 3). Each paragraph is accompanied by a large number of exercises; thus, there is enough material for both classroom work and work at home.

The textbook's central concept is the mathematical model. For example, the derivative is introduced as follows. After examining two problems that are standard in this situation — one on instantaneous velocity and one on tangents — the authors state:

> Two different problems have led us to the same mathematical model — the limit of the ratio between the change in a function and the change in its argument, on the condition that the change in the argument approach zero.... This mathematical model, then, is what should be studied. That is:
>
> (a) It should be given a formal definition and labeled with a new term;
> (b) New notation should be introduced for this model;
> (c) The properties of this new model should be investigated. (Mordkovich and Smirnova, 2009a; p. 232)

The distinctive feature of the presentation of the topic "Derivatives" in this textbook consists in the fact that it begins with the presentation of the limit of a sequence. This concept is defined in the language of "neighborhoods" and explained in a sufficiently detailed and clear fashion. The limit of a function is first introduced at infinity, and only afterward at a point; in neither case is a formal definition given. In general, there are relatively few proofs here. For example, the paragraph on the "Rules of Differentiation" is structured as follows. First, the textbook formulates four theorems concerning the derivative of a sum, the derivative of the product of a function and a number, the derivative of a product, and the derivative of a quotient, and provides examples. The authors then write:

> First, we will derive the first two rules of differentiation — this is relatively easy. Then we will examine a number of examples of the ways in which the rules and formulas for differentiating are used, so you can get used to them. At the very end of the paragraph, we will give a proof of the third rule of differentiation — for those who are interested. (Mordkovich and Smirnova, 2009a, p. 244)

The conditions for the monotonicity of a function are illustrated using a physical interpretation; the theorem concerning necessary conditions for the existence of an extremum, usually referred to in Russian textbooks as Fermat's theorem, is not proven (nor is it referred to by the name of its author). In general, the textbook contains practically no historical information. In this way, it is oriented more toward practice than theory. Possibly, this accords with the idea of teaching mathematics on the basic level.

6.4.2 *The textbook of G. K. Muravin and O. V. Muravina*

In addressing students in the foreword to this textbook, the authors emphasize: "To know mathematics means to be able to solve problems. It is problems that you will have to solve on the Uniform State Exam" (Muravin and Muravina, 2010b, p. 5). Despite this declaration, the textbook devotes considerable attention to theory and to working with concepts and theorems. The concept of continuity is introduced at first on an intuitive level: the graph of a continuous function can be drawn without lifting pencil from paper. Using this visual image of continuity, the authors next introduce the interval method for solving inequalities. In 11th grade, the definitions of continuity and the limit are introduced in the language of "ε–δ." Quantifiers are used in the formulations of definitions. Problems that involve computing simple limits are solved. Theorems on the limits of sums, products, and quotients are formulated, but not proven; it is pointed out, however, that they "may be proven, and even without much difficulty" (Muravin and Muravina, 2010b, p. 25). The textbook examines vertical, horizontal, and oblique asymptotes to the graphs of functions.

In connection with the introduction of derivatives, the concept of the tangent is raised and discussed first, followed by derivatives and differentials. The derivatives of elementary functions are introduced in the same way as they are in Kolmogorov's textbook: in connection with geometric considerations, the number e is introduced as the base of the exponential function e^x, whose derivative at zero is equal to 1; then the derivatives of exponential, logarithmic, and power functions are introduced. In the presentation of integral calculus, the authors first examine the area of a curvilinear trapezoid, then introduce the integral as the limit of integral sums, and then demonstrate that the derivative of a variable area is equal to the function $f(x)$; only after this do they bring in the concept of the antiderivative.

On the whole, the textbook combines a sufficiently high theoretical level with clear explanations, a well-phrased presentation, and a large number of historical discussions. At the same time, it contains many problems and devotes considerable attention to methods for solving them.

7 Conclusion

In arguing for the need to teach the elements of calculus in school, educators usually refer to the fact that, first, it is desirable for students to have some knowledge of calculus at the outset of their college education, if only because without such student knowledge it is difficult to teach other courses. Second, they argue that calculus is one of mankind's most important intellectual achievements, and it is desirable that even those who will not go on to study mathematics in college acquire some understanding of it. As already noted, arguments can also be made against this position, relying on the experience of other countries in which relatively few study calculus. No one, apparently, denies the need to develop students' ability to think in terms of functions (which can be done even without calculus), but even here quite different approaches are possible.

In Russia, calculus has been taught to students in the highest grades, in one form or another, for almost a half-century. But while the content of what students are taught has remained relatively stable, the manner in which they should be taught remains a subject of debate. Different opinions exist about the degree of rigor with which various propositions must be proven, whether or not a formal definition of limit is required, what quantity of geometric and physical applications should be examined, and whether attention should be primarily focused on theory or practice. We should note, however, that even the most "proofless" and "recipe-like" textbooks still operate under the assumption that proofs must be carried out (even if outside of school).

While focusing on an overview of textbooks, we should not lose sight of the fact that actual education takes place in the classroom. A teacher who has not understood the subtle arguments of a textbook's author which delight experts can do much harm to students. Convesely, a highly qualified teacher can contribute to a "recipe-like" textbook, using it only as a kind of reference manual for the students and as a problem book. On the other hand, it is evident that teachers by and large are trained by the textbooks which they use in teaching. Thus, if a textbook straightforwardly states that there is no need to prove or even to explain, then more than a few teachers will become accustomed to this idea and even extend it, discovering that not only is there no need to explain derivatives, but that there is no need to explain anything at all.

At present, schools are offered a wide variety of textbooks; formally, they can choose from a long list. However, a textbook's quality is not always the determining factor in this selection. A great many other circumstances can play a decisive role: the traditions of the school and the district, habit, or even the fact that the school library already has a large number of certain textbooks.

Furthermore, practical considerations limit the use of technology in the study of functions, which foreign readers have probably already considered more than once while reading this chapter. In contrast to what has happened in other countries, in Russia, the graphing calculator has thus far not become an everyday instrument for every student. It may be supposed that its appearance will usher in certain changes — although, as we have seen, for example, the idea of presenting various functions, and not just linear ones, to students at very early stages of schooling can be implemented without calculators; and, conversely, the presence of a calculator by no means guarantees that students will not assume that all functions are necessarily linear.

In any event, in summing up, it may be said that Russian educators have accumulated extensive experience with teaching calculus as part of mass education, including techniques for presenting theoretical material and interesting problems. The teaching continues, and experience — both positive and negative — continues to accumulate.

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6

Combinatorics, Probability, and Statistics in the Russian School Curriculum

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1 Finite Mathematics in the School Curriculum Prior to the Revolution of 1917

The debate over the role of statistics and probability theory in the school curriculum goes back as far as the first half of the 19th century. Interest in these subjects was informed in large part by the significant contributions made to the field by Russian mathematicians: at the time, foreign scholars jokingly referred to probability theory as the "Russian science."

By the mid–19th century, N. T. Scheglov, an instructor of algebra at the Tzarskoselsky *lycée*, published a textbook covering several topics in probability theory: "Simple or absolute probability. Conditional probability. Complex probability. Probability of interchangeable events. Probability of events in repeated experiments." (Scheglov, 1853). The text covered the basic principles of these topics and offered sample problems and solution strategies.

The popular textbook for elementary algebra by K. D. Kraevich (1866) — as well as his exercise book (1867) — included chapters "on probabilities." Kraevich set out the material in an informal manner, typically emphasizing practical application and avoiding rigorous proofs and formulas (for example, "Mathematical advantage. On the lottery. On the probability of human life. On insurance."). A detailed study of the probability curriculum, as envisioned in the textbooks of the period, may be found in A. Kolmogorov's (1947) work on the role of Russian science in the development of probability theory.

The plans and methods of teaching probability theory in the secondary school were actively debated in the early 20th century, as part of the larger discussion of reforms in mathematical education. It should be noted that at that time, curriculum reform — particularly integration of probability theory into the general course of study — was felt to be a necessity not only among mathematicians but also in the natural sciences. A model syllabus in probability theory for the secondary school, developed by P. S. Frolov, was published in 1902 in the *Proceedings of the XI Conference of Russian Naturalists and Physicians*. By the XIII conference, delegates were considering two distinct curricula, incorporating an introduction to probability and statistics: basic-level and advanced-level.

A detailed account of the proceedings, along with a chronicle of events and related documents, may be found in a publication released by the Ministry of Education (Ministry, 1915). The plan of integrating probability theory and the closely associated combinatorial analysis into the secondary school algebra curriculum was raised once more in a report delivered to the Ministry of Education by a special commission charged with curriculum reform in mathematics, and debated at teachers' conferences. The Ministry gave the matter serious consideration, soliciting the opinions of teachers and professors as well as expert analysis. The ensuing publication remarks on "general educational advantages of learning to calculate probability with combinatorial analysis, as well as its practical applications in the areas of trade, financing, management, and accounting)" (Ministry, 1915, p. 48). Beyond educational benefits, the outlined curriculum was also seen as an important formative tool. Consequently, the Ministry was interested in the opinions of experts from a variety of disciplines.

Another innovation discussed at pedagogical conferences was the establishment of a two-tier secondary school curriculum with a general track and an advanced track "adapted to a variety of individual learning abilities and sensitive to the needs of educated people." Accordingly, the commission considered two proposals for a two-hour course and a four-hour course in probability, the first submitted by one of its members, Professor P. A. Nekrasov, and the other by the director of the Uriupinsk real school, P. S. Frolov (Ministry, 1915):

Frolov Plan	Nekrasov Plan
I. A two-hour basic course:	
1. Combinations	1. Combinations
2. Introduction to probability	2. Introduction to probability
3. Newton's binomial theorem	3. Newton's binomial theorem
4. Bernoulli's theorem	4. Bernoulli's theorem
5. Statistical correlation	5. Transformations of Bernoulli's theorem
II. Additional topics included in the four-hour course:	
6. Multiplication of probabilities	6. Multiplication of probabilities
7. Addition of probabilities	7. Addition of probabilities
8. Huygens' problem	8. Huygens' problem
9. Bayes' theorem	9. Comparison of statistical arithmetic means and mathematical expectations; Chebyshev's theorem of means; Statistical correlation
10. Witness testimony	10. Bayes' theorem
11. Buffon's problem	11. Witness testimony
12. Gambler's ruin problem	12. Buffon's problem
13. Mathematical expectation	13. Gambler's ruin problem
14. Life insurance	14. Additional topics in mathematical expectation; price
	15. Life insurance

It is evident that Russian scholars were largely in agreement: the study of probability begins with combinatorial analysis, followed by additional topics in statistics, historical aspects, and practical applications. The two versions of the basic course comprise essentially the same elements, differing primarily with respect to the sequence of presentation. The two-hour course would be integrated "into the secondary school general curriculum. The new course may be accommodated into the curriculum by eliminating certain less significant theories and the more abstract and futile exercises" (Ministry, 1915, p. 35). The four-hour course was intended for the newly organized "real schools" (or modern-language schools, as they were also called), offering courses useful for the future study of economics, biology, or other subjects dependent on the descriptive or comparative inductive method and grounded in mathematical statistics and probability theory.

Regrettably, the planned integration of probability theory into the secondary school curriculum — first experimentally and then on a mass scale — was never realized because of historical circumstances: the breakout of the First World War, followed by the October coup, which ushered in a new state, soon to be known as the USSR.

2 Finite Mathematics in the Secondary School Curriculum in the Soviet Period

Proposals for integrating probability and statistics into the secondary school curriculum were frequently debated in the Soviet era. The explanatory statement accompanying the curriculum plan for the secondary stage of the Unified Labor School–Commune, as well as subsequent similar statements, discussed the necessity of including probability theory in the course of study, pointing out the widespread application of the statistical method in contemporary physics (The Second Stage, 1929).

Such topics as probability of events, addition and multiplication of probabilities, the law of large numbers, elements of mathematical statistics, and the law of random errors were included in curriculum proposals throughout the 1920s. Debates over methodology continued at this time in the professional pedagogical press, featuring curriculum proposals and strategies for their implementation.

Demand for a variety of practical applications of probability theory also continued to grow at this time. It is no accident that the classic text *Basic Introduction to Probability Theory*, by B. V. Gnedenko and A. Ya. Khinchin (a title that makes the book's intentions perfectly clear), was written during the Second World War (first edition in 1945). Recognizing that "a large number of leaders (and even some employers with less responsibility) — be it in the military, industry, agriculture, or economics — must often make use of the science of probability" (Gnednko and Khinchin, 1945). Aiming to make up for a lack of preparation, the authors set out the fundamental principles of stochastics at the very basic level of mathematical knowledge, emphasizing practical application of probability and statistical laws without entering into specialized and formal matters.

At the time of the educational reform of the 1960s, the leading Russian mathematicians — A. N. Kolmogorov, A. Ya. Khinchin, B. V. Gnedenko, A. I. Markushevich, and I. M. Yaglom — called for the integration of topics in probability into the general course in mathematics. Gnedenko (1968) wrote:

> The graduating young citizen must be well aware of the fact that very few social and natural processes are reducible to pure causality. The next step on the road to knowledge is the statistical approach. Herein lies the tremendous methodological significance of statistics and probability theory. […] In the interest of advancing all branches of science we must incorporate elements of statistical analysis into the school curriculum.… (p. 11)

The great mathematician and pedagogue A. N. Kolmogorov devoted a series of articles to the problems of integrating topics in "contemporary mathematics" — including statistics and probability — into secondary school education. He developed an elective course for high school students that included the study of probability and devised a variety of teaching strategies (Kolmogorov, 1968, pp. 63–72).

In a series of books and articles appearing around the same time, the renowned mathematician I. G. Zhurbenko (1972) attempted to formulate the main ideas and applications of probability theory at a level accessible to secondary school students. N. Ya. Vilenkin likewise published several texts in combinatorial analysis (e.g. 1969), offering in-depth analysis of an assortment of concrete problems varying in difficulty. At the same time, a number of scholars published works aimed at formulating the methodology for teaching the new course. Some of the methodologists called for an independent course dedicated strictly to the study of the principles of probability theory (e.g. Gaisinskaya, 1972; Potapov, 1969; Veliev, 1972), while others argued for a combined combinatorics–probability curriculum (Dograshvili, 1976; Kabekhova, 1971; Samigulina, 1969).

The majority of proposals leaned heavily toward probability, with very limited space given to the elements of statistics. At the same time, the initiative for integrating probability theory and statistics into the secondary school curriculum was actively promoted not only by scholars and teachers of mathematics but also by physicists, chemists, and biologists. The need for a preparatory course in probability and statistics in the secondary school was also discussed at the college level. In a discussion on "equally likely events," E. S. Venttsel, author of one of the most popular college textbooks on probability theory, spoke about "events, not reducible to a system of chance occurrences," stressing that "all of these techniques are grounded in experiment, and in order to master them one must first learn about frequency of event and grasp the organic connection between probability and frequency" (multiple editions, e.g. 1998, p. 9). The author noted that college students have a difficult time absorbing the principles of probability and statistics without preparatory work at the secondary school level.

Despite all this, a course in probability was not included in the finalized version of the secondary school curriculum. A. N. Kolmogorov (1968), the founding father of Russian probability and mathematical statistics, expressed his regret in the following words: "Unfortunately, no positive solution could be found to the problem of integrating elements of probability theory into the secondary school curriculum" (p. 22). Pilot programs had shown overwhelmingly that teachers of mathematics and the school system as a whole were unprepared to take on the new and unfamiliar subject. It should be noted, however, that arguments of this sort are always valid, and always stand in the way of genuine reform. At the same time, it must be acknowledged that the failure of this particular reform was due in large part to the emphasis by pilot programs on a theory-heavy approach to probability, i.e. the classical *a priori* approach to the notion of probability of a random event, at the expense of practical application and interdisciplinary implications. As a result, probability theory was almost completely cut off from mathematical statistics, the latter being entirely omitted from the course. In the experimental textbook for the ninth grade edited by Kolmogorov, the section on probability theory followed directly after — and elaborated upon — the section on combinatorics. Consequently, the two sections made up a peculiar fragment, disconnected from other topics in the course and from other subjects in the curriculum, thus failing to attract the interest of practically minded 15- and 16-year-old students and their teachers. Regrettably, the attenuated approach of Kolmogorov's textbook had little in common with the methodologies elaborated by the master pedagogue in his writings, and did much to discredit the very idea of integrating probability into secondary school curricula.

As a result, combinatorics and elements of probability theory were cast out to the educational periphery, i.e. high school electives or courses in schools that specialized in mathematics, where these subjects were taught at the very end of the final year. Here, too, they suffered from the theory-heavy approach: rather than seeking a deep and intricate understanding of the probability of a random event or honing their skills in mathematical modeling, students in specialized physics and mathematics schools were asked to solve complex probability problems by virtue of increasingly complicated combinatorial analysis, a rapid and formal shift toward conditional probability, and Bernoulli's formula. Moreover, the foundations of probability were taught with no reference to mathematical statistics. Finally, the isolated and strictly theoretical fragment that comprised the elements of probability theory never became a full-fledged component of the curriculum, even in specialized schools, as evidenced by a nearly total absence of probability-related problems on final examinations in classes with an advanced course of study in mathematics.

In this context, the decision to include problems on probability in experimental final examinations for schools with advanced courses of study in mathematics, made in the 1990s by the St. Petersburg examination board chaired by A. P. Karp (see Karp, 1997), seems almost heroic. However, even these problems were inevitably formal and, at the same time, relatively basic from a theoretical probabilistic perspective. They required little more than "plugging" data into a formula, which seems to run counter to the standards of advanced courses and suggests that the compilers of the exam were unsure of students' abilities to confront the probability in any real depth. Here is a sample problem from those examinations:

> A complex number z is chosen at random, such that $|z| = 1$. What is the probability that $|z - 1| \leq 1$? (Karp, 2000, p. 162)

Even earlier, a variety of researchers and instructors, concerned with promoting statistical thinking in secondary school students, had developed teaching materials and conducted experiments with extracurricular or elective courses in statistics (Avdeeva, 1970; Ochilova, 1975). However, the limitations of such a platform and the voluntary nature of these courses ran counter to the very objectives set out by the authors: to promote in all students the basic principles of statistical thinking, indispensable in a variety of fields outside the mathematics class.

Subsequent attempts to integrate stochastics into the curriculum were largely based on the work of V. V. Firsov (1970, 1974), who demonstrated that development of statistical thinking and probability intuition demands a practically oriented course. Firsov asserted that the study of probability should include such steps of applied problem-solving as formalization and interpretation. Nevertheless, despite numerous convincing arguments and reasoned conclusions, the problem of bringing probability into the classroom could not be solved, as Firsov himself acknowledged, without extensive on-the-ground testing of methodological ideas and techniques.

In all fairness, it must be acknowledged that although Russia had until recently remained virtually the only nation in the developed world where probability and statistics were omitted from the secondary school curriculum, the country's scholars, methodologists, and teachers continued all along to test a variety of approaches to teaching the foundations of these sciences. A series of experiments in adapting and advancing the methodology of teaching probability and mathematical statistics were staged in the 1970s and 1980s in the USSR.

An interesting interdisciplinary experiment was conducted about this time by K. N. Kuryndina (1980): according to Kuryndina's schema, several topics in probability and statistics were covered in mathematics courses, while others were covered in geography, elective courses or mathematical circles (clubs). This experiment was further developed by V. D. Seliutin (1983, 1985), in the city of Orel: here, too, a comprehensive course of study in stochastics was divided among a variety of mathematics courses, optional courses, and circles. These experiments demonstrated the accessibility of the material — when oriented practically, its powers of promoting statistical thinking, as well as the students' interest in a practically oriented course in stochastics. Seliutin's approach is distinguished by its emphasis on statistics and decision-making in real-life situations. This localized experiment also showed that topics in probability are accessible — and useful — to students as early as in middle school.

This conclusion was supported by L. O. Bychkova (1991), who demonstrated that teaching probability and statistics in the fifth and sixth grades was both feasible — from a psychological-pedagogic perspective — and productive. Bychkova's research focused primarily on the development of statistical thinking. In the fifth grade, the study of probability took up 10 hours, of which half was spent on combinatorics and the other half on statistical data (data grouping, arithmetical mean, bar charts). In the sixth grade, 15 hours were spent on probability, of which 8 were taken up with the study of the theory of probability proper (experiments with random outcomes, random events, certain and impossible events, classical definition of probability of a random event, solving problems on probability, frequency and probability), while the other 7 went to basic statistical analysis (statistical data, mode and range of sample, statistical analysis).

To test the development of statistical thinking among students, Seliutin and Bychkova made use of qualitative problems proposed by V. V. Firsov (1974) and analogous problems geared to other age groups. These problems were given to students who had covered the elements of probability theory and statistics, as well as to students who had not covered these topics. This experiment confirmed the hypothesis that the study of probability as a subset of pure mathematics based on the classical definition of probability has no significant effect on the development of statistical thinking among students and is perceived by the students as a topic in pure mathematics without practical application.

3 Finite Mathematics in the Post-Soviet Period

The virtual absence of probability and statistics from the Soviet curriculum was due neither to chance nor strictly to internal methodological and pedagogical problems. We may recall that practically all statistical information in the Soviet Union was either marked "classified" or available in a truncated, distorted, ideologically "purged" interpretation. In this environment, the teaching of statistics and probability in the secondary school must have appeared to the powers that be not only unnecessary but also ideologically harmful.

It is no accident that a wave of renewed interest in integrating probability and statistics into the curriculum in the late 1980s to the early 1990s coincided with the collapse of the Soviet system, a period of Gorbachev democratization and *perestroika*. The need to move from localized and uncoordinated experiments to a mass-scale experiment and subsequent integration of stochastics into the mandatory mathematics course was voiced at two international conferences on the teaching of probability and statistics in secondary school, convened in the 1990s by the recently created Russian Association of Mathematics Teachers (for a detailed account, see Bulychev, 1996). Previous experiments in integrating stochastics into the curriculum were shown to be inconsistent and unfocused, and these problems were discussed in the context of international practice.

Simultaneously, a group of educators, including the present author, from the Laboratory for Mathematical Education at the Institute on Educational Content and Methods of the Russian Academy of

Education, brought out in 1994 and subsequent years an instructional "set" for the fifth-to-ninth grades (the textbooks *Mathematics 5* and *6*, edited by G. V. Dorofeev and I. F. Sharygin, and *Algebra 7*, *8*, and *9*, edited by G. V. Dorofeev). For the first time, statistics and probability were given equal footing with more traditional topics (i.e. number, expressions, functions, equations, and inequalities). The material was carefully divided up among the years 5–9, with emphasis on the concrete and practical aspect. Because the "set" was included in the Federal Registry of textbooks recommended for use in Russian schools, it received substantial exposure across the country. Moreover, although at that stage of curriculum implementation sections dealing with statistics and probability were not considered mandatory, we may still speak of a relatively mass-scale experiment. The experiment yielded positive results, and this, along with other arguments, ensured that statistics and probability remained under consideration at every stage of the fierce debates on educational standards in mathematics, and was finally included in the new Russian Educational Standard (Ministry, 2004). By order of the Ministry of Education, the general integration of stochastics into teaching practice began at this time. A memo sent out by the Ministry in 2003, titled "On the implementation of elements of combinatorics, statistics and probability theory into the general curriculum," proposed that a pilot integration program might begin as early as school year 2004–2005 (Ministry, 2003).

Since, as yet, no curriculum for a high school course in stochastics has been tested in a large-scale trial program, we will cite here an excerpt from a section of the aforementioned Standard of Basic Education in Mathematics pertaining to combinatorics, statistics, and probability, followed by an analysis of proposed methodology and examples:

> *Combinatorics.* Problem-solving strategies: enumeration of variants, product rule.
>
> *Statistical data.* Representing data in tables, diagrams, graphs. Means of data received by measurement. The principle of statistical inference based on sampling. Definition and examples of random events.
>
> *Probability.* Frequency of event, probability. Equally likely events; finding their probability. The principle of geometrical probability.

PERFORMANCE REQUIREMENTS FOR GRADUATING STUDENTS

At the end of his or her studies, a student must

know/understand:

- the stochastic basis of a wide range of natural phenomena; examples of statistical regularity and statistical inference;

be able to:

- interpret information presented in tables, diagrams, graphs; generate tables, diagrams, graphs;
- solve combinatorial problems by the method of enumeration of variants as well as by using the product rule;
- calculate the mean value;
- find event frequency from direct observation or supplied statistical data;
- find probability of random events in basic situations;

be able to deploy acquired knowledge and skills in concrete everyday activities:

- analyzing practical numerical data presented in the form of diagrams, graphs, tables;
- solving practical problems in everyday and professional activity involving numbers, percentages, length, area, volume, time, velocity;
- solving real-world and school problems using the method of systematic enumeration of variants;
- comparing probabilities of random events, evaluating the probability of random events in real-life situations, contrasting models with real-life situations;
- interpreting statistical assertions.

We can judge from this excerpt that the architects of the new curriculum wished, at this introductory stage, to limit the course in statistics and probability to the basic principles and notions, while conferring upon it a sense of unity and comprehensiveness. It had also been decided at this stage to omit such concepts as “conditional probability” and “mathematical expectation,” along with several others, which had proven too complex for the majority of students and even teachers during the mass-scale pilot program.

At this time, the course in stochastics may be divided into three major components: combinatorics, elements of probability theory, and elements of statistics. While these remain autonomous subjects, their interaction is expected to produce the results outlined above.

In the course in stochastics, combinatorics plays a somewhat secondary role. However, while until now it has been firmly confined to courses for the gifted, electives, summer courses, tournaments, and Olympiads, here the foundations of basic, preformula combinatorics are for the first time integrated into the general curriculum and made available to all students. As a result, each student will learn the method of enumeration of possible variants, and be able to identify and use various possible orders of this enumeration: ascending, alphabetized, tree-diagram, and so on, which will be used in calculating the number of favorable and all possible outcomes in solving basic problems for calculating *a priori* probability in a classical schema. This knowledge is also required for subsequent study of the foundations of descriptive statistics, used in the classification of objects based on given parameters. It should be noted that students are expected to use their new skills in logical enumeration and combinatorial thinking in connection with other topics such as visual geometry, number divisibility, and word problems.

With the study of probability, the emphasis is on promoting probabilistic thinking matched with the students' abilities at specific age levels. The students are exposed to the classical and statistical approaches to the concept of probability, which are meant to be complementary and mutually informative. Without this balance, the students are invariably left with a limited and skewed understanding of probability. At the same time, the frequency approach (statistical approach) is somewhat more emphasized. The teaching of the classical approach (based on the hypothesis of equiprobability) as foundational has had largely negative results in Russian schools, which finally led to the exclusion of probability from the general mathematics curriculum during the reform of the 1960s. Emphasis on the classical approach leads invariably and quickly to operations with complex formula combinatorics and the principle of conditional probability, which are beyond the psychophysical and intellectual capacities of the average secondary school student, while continual reliance on the hypothesis of equiprobability leads to distortions and errors when students begin to consider real-life situations.

The frequency (statistical) approach, while not free of certain methodological problems, also possesses a number of advantages, especially at the early stages of the study of probability. The essential preparatory course involves direct observation, experimentation, and discovery of concrete, observable patterns in random events. The presentation of the material is carefully paced, allowing students to become familiar in due time with the classical definition of probability and principles of geometrical probability, and preparing them for a smooth transition — in college or advanced high school courses — to the axiomatic approach to the concept of probability.

Within the proposed framework, statistics becomes the central component of the entire stochastics curriculum, as outlined in the Standard. Meanwhile, the required volume and difficulty of the material are dictated not only by the general aim of promoting probabilistic thinking in students, but also by the need to solve basic statistical problems — just as the required volume and difficulty of the material in combinatorics are dictated by the need to provide students with the mechanism for calculating basic probabilities.

One of the most important objectives of the study of statistics is active participation of students in the general process of statistical investigation, which brings together into a single unified whole the formulation of key principles; the process of gathering and sorting data; its subsequent plotting in the form of tables, diagrams, and graphs; and subsequent analysis of these data and interpretation of results. Since practical application is the primary goal in the study of statistics in secondary schools, it is imperative that students gain an understanding of the meaning of statistical predictions and conclusions, to be found in all aspects of social existence: from television commercials and sports betting to political and social predictions and commentary.

The teaching of statistics presupposes continuous reference to real-life statistical data, public opinion polls, and other number-driven activities, working with practical, applied problems, and providing reasoned interpretations of results. Moreover, the natural placement of stochastics in the mathematics curriculum requires a gradual and timely transition from descriptive preparatory procedures and consideration of the notions at a qualitative level to the study of quantitative stochastic correlations, corresponding to a level of formalization dictated by specific age-level requirements and continuity of presentation and taking advantage of intradisciplinary connections (tying stochastics with the study of percentages, ratios, ordinary fractions, working with graphs, calculators, etc.).

At the time of the experiment, only one textbook "set" for the general secondary school incorporated stochastics (Dorofeev and Sharygin, *cf.* above); today, there is a boom in publishing and integrating a variety of practical study materials, textbook supplements, and study aids containing combinatorics, probability, and statistics material.

One of the characteristics of these publications is that they all address the pressing issue of the day: formulation of a methodology for the practical integration of stochastics into the general curriculum. At the same time, they are frequently written outside the context of theoretical research, and of the contradictory and largely negative theoretical and practical experience of past attempts at integrating stochastics into our schools. Moreover, they do not take into account international practice and developments in teaching stochastics, and often ignore even regulatory guidelines or, simply, the extremely limited number of hours allotted at this stage to the study of the foundations of stochastics in school.

Once more, we find the study materials offered for general implementation full of dogmatism, overloaded with content, and containing the idiosyncratic ideas and biases of their authors. Curricula incorporating stochastics plot unwieldy courses of study, poorly suited to students' age levels and intellectual capacities, and quickly shifting toward abstraction while adding standard university courses in probability to the school curriculum (Dyadchenko, 1994; Fedoseev, 2002).

Among the undisputed advantages of the recently published material (Makarychev and Mindyuk, 2003; Mordkovich and Semenov, 2002; Nikolsky *et al.*, 1999–2001; Tkacheva and Fedorova, 2004), in addition to providing prompt practical response to time demands, is the attempt to integrate new material into an existing course and into mathematically and practically interesting problems. We should note that along with these textbook "sets," a number of study guides for the fifth-to-ninth grades are likewise attempting to address the pressing issues of the day, including *Probability and Statistics* and *The Foundations of Statistics and Probability*, published by the present author in collaboration with V. A. Bulychev (2002, 2004), as well as the study guide of Yu. N. Tyurin *et al.* (2004).

In connection with ongoing efforts to integrate topics in probability theory, combinatorics, and statistics into the new Russian standards for mandatory mathematical education, *Matematika v shkole* — virtually the only existing domestic journal addressing the problems of teaching methodology in mathematics — published a special issue in 2009 devoted to the content, methodology, and possible monitoring of the new course in stochastics. In an article featured in that issue, the present author laid out the following objectives for integrating the foundations of probability and statistics into the general curriculum (Bunimovich, 2009, p. 31):

1. Acquiring command of a system of probability and statistical concepts, indispensable in everyday existence, for the study at a contemporary level of social and natural sciences in the secondary school, as well as in the advanced stages of academic or professional education.
2. Acquiring an understanding of the universality of the laws of probability and statistics, of stochastics as the foundation of the contemporary description of the scientific worldview, and as a tool for modeling social, economic, and natural processes and phenomena.
3. Developing a probabilistic intuition, statistical culture, combinatorial thinking, and ability to draw substantiated conclusions from available data.
4. Becoming familiar with such crucial methods of inquiry as finding patterns in random processes, constructing adequate

models of phenomena, and testing hypotheses with experiments.

5. Enriching the personality through the discovery of philosophical aspects of concepts in statistics and probability by studying the history of their development.
6. Fostering genuine patriotism by way of considering the contribution of Russian scientists to the development of probability theory and mathematical statistics as a full-fledged branch of mathematics, and recognizing the achievements of native mathematical science as part of the national heritage.

4 Features of Contemporary Approaches to the Study of Finite Mathematics in Russian Schools

The integration of combinatorics, probability, and statistics into the general mathematics curriculum "for all" and the outlined objectives imply a change in the traditional approach to teaching the subjects. Special attention must be paid to the preparatory stage, where students are first introduced to the concepts of combinatorics, probability, and statistics. Below, we will consider approaches proposed by the present author (1994–1997) and largely endorsed by the authors of the major textbooks in use today, as can be seen from their collective article (Bunimovich, Bulychev, Tyurin, Makarov, Vysotsky, Yaschenko, and Semenov, 2009) setting out the general approaches to teaching the new material.

The combinatorial component of the stochastics curriculum is the most familiar of the three to the audience in terms of methodology and teaching structure, and it comes closest to the traditional system of material presentation. Once again, the most novel addition is the preparatory stage, i.e. visual *preformula* combinatorics.

To solve problems in combinatorics, students aged 10–13 first of all use the method most natural and accessible to their age group: systematic enumeration of variants. At the preparatory stage, solving a problem in combinatorics means writing out all possible combinations of numbers, words, objects, etc., as the problem requires. This type of exercise teaches students the usefulness of enumeration of different kinds of combinations.

This approach to combinatorics, as opposed to the "formula-based" approach, allows the class to consider a far broader range of combinatorial problems, not limited to the basic formulas for the number of possible permutations and combinations, but also including combinations with repetitions and enumeration with different restrictions, while at the same time allowing a shift in focus to the most difficult part of solving problems in combinatorics: formalization of the problem and construction of a suitable model.

Different organizational methods for systematic enumeration are covered — e.g. ascending (number), alphabetical (letter) — as well as enumeration using a special graphing technique: a tree of possible variants, which provides a convenient starting point for systematic enumeration. Diagrams and coding techniques not only simplify notation but also touch on some of the essential issues in mathematics, such as mathematical modeling and universality of mathematical techniques.

The proposed methodology may be better understood through an example of coding and enumeration of possible variants drawn from the textbook *Mathematics 6* (Dorofeev and Sharygin, 1997, p. 249):

> Eight friends meet and all shake hands. How many handshakes did the friends exchange?

To solve this problem, students use a two-step coding process. First, every friend is assigned a number from 1–8. Then, every handshake can be coded as a two-digit number, made up of numbers from 1–8. It is important that the meaning of this "coding" is not lost in the process; for example, the students must understand that the number 47 denotes the handshake between the fourth and the seventh friend. It is important to explain why a handshake code 33 is impossible in this situation: it would mean that one of the friends shakes his own hand; or that the codes 68 and 86 denote the same handshake, and consequently only one of these numbers must be counted (for example, only the smaller). Next, the students are asked to count all possible two-digit numbers, composed of numbers from 1–8, where the first digit is smaller than the second. It makes sense to write them out in an ascending order, which yields the following "triangle," giving us the total number of handshakes:

12, 13, 14, 15, 16, 17, 18,
23, 24, 25, 26, 27, 28,
34, 35, 36, 37, 38,
45, 46, 47, 48,
56, 57, 58,
67, 68,
78.

Comparison of various methods of enumeration proposed by different students in their approach to the same problem — including image-based approaches, such as a "tree" of possible variants, either sketched or imagined, as well as logical arguments — activates the child's imagination and logical thinking. When learning about the systematic method of enumeration, emphasis is given to choosing the most rational coding strategy and the most convenient method of enumeration.

The next step is familiarizing the students with the rule of product, fundamental to the classical formulas of combinatorics: the formulas for the number of permutations and combinations. This happens naturally with the transition from problems with relatively few items, permitting exhaustive enumeration, to problems with large numbers of variants, where constructing a "tree" or using any other method of direct enumeration proves technically inefficient.

Let us note that the general methodological goal lies in eliciting the conceptual basis of the problem and finding an appropriate mathematical model. This is necessary in order to avoid the typical pitfall of the "formula-based" approach to combinatorics: the temptation to "plug in" rules and formulas mechanically. To prevent students from developing the incorrect stereotype, sets of problems using the rule of product will typically include several problems where the straightforward use of multiplication will not yield the correct answer. This exposes the limits of the rule's application and keeps mindless formalization at bay. Here is an example of such a problem, analogous to the “handshake problem” (p. 258):

> Sixteen players take part in a chess tournament. Every player will meet every other player in a single match. How many matches will be played overall?

When encountering this problem, students may reason as follows:

> Each match involves two players. The first of these may be any one of the 16 players; the second may be any one of the remaining 15. By using the rule of product, as in several previous scenarios, we arrive at the total number of matches: $16 \cdot 15 = 240$.

However, in this case, each of the matches was counted twice: once counting all the matches played by the first player, and again counting all the matches played by the second player (the teacher may use a tournament table to illustrate this point). In reality, half the matches were played: $\frac{16 \cdot 15}{2} = 120$.

The preparatory stage, where the material is presented at the visual and qualitative level, is likewise the radically new addition to the Russian teaching of probability in terms of both content and methodology. At this stage (see Bunimovich, 2009; Tyurin *et al.*, 2009), students are encouraged to study and actively investigate stochastic situations and processes. To this end, classes engage in group discussions on various classroom exercises and experiments, and work together on constructing probability models. The students must consciously apply the results of the experiments to analysis and prediction. This strengthens their motivation to understand not only the principles of stochastics but also related concepts belonging to other branches of mathematics (proportions, parts, fractions, percentages, graphs, areas of geometrical figures, etc.).

New challenges arise when students encounter problems where chances of such-and-such an event may not be determined with precision but must be approximated, based on life experience, previously derived statistical data, or a series of experiments. Because the probability of an event is contingent on the circumstances in which it is examined, several answers offered in a class discussion may prove correct — something that is unexpected and unfamiliar not only for the students, who have by now developed certain stereotypical notions about the learning process, but often even for the teacher.

Let us examine a solution strategy for a problem from *Mathematics 5* (Dorofeev and Sharygin, 2001) found in the teacher's guide for the textbook (Suvorova *et al.*, 2001, p. 92):

> Using personal experience, evaluate the chances of the following random events and determine which would be the most probable:
>
> (a) No one will call you between 5 am and 6 am;
> (b) Someone will call you between 5 am and 6 am;
> (c) Someone will call you between 6 pm and 9 pm;
> (d) No one will call you between 6 pm and 9 pm.
>
> Problems of this sort expose your students to general statistical patterns as well as to personal peculiarities, which will result in differences of individual answers to the same question. Because phone calls are generally rare early in the morning, chances of (b) are extremely low, it has negligible probability — a practically impossible event; whereas (a) is highly likely — it is practically a fact. The evening hours are, on the contrary, a time of high "telephone activity"; thus, for most people, option (c) will be more probable than option (d); although if a person generally receives very few phone calls, (d) may turn out to be more probable than (c).

As has been said already, one of the main features of the adopted methodology is the statistical approach to the concept of probability, as the most immediate and grounded in the students' experience. The probability of a random event is evaluated with respect to its relative frequency, which is derived from empirical data. This approach requires students to gather the necessary data as part of the learning process. Moreover, to stabilize frequency, an experiment must be repeated a sufficiently high number of times.

The staging and conducting of experiments is an integral part of solving problems in probability. At the first stage, these are actual experiments with real objects. At later stages, students are expected to model experiments with random outcomes using a computer.

Real-world application is likewise the leading aspect of the statistical component. To illustrate this, let us examine two sample problems/mini-investigations from the textbook *Mathematics 9* (Dorofeev, 2000, p. 308):

1. It is known that "o" is the most commonly used vowel in the Russian language. Read over the following excerpt from the poem *The Bronze Horseman*, by Alexander Pushkin [a commonly anthologized excerpt, beginning with the lines "Upon the shores of desolate tides ...."].
   (a) Does this excerpt confirm the claim made at the start of the problem?
   (b) Compare the relative frequency of the [cyrillic] letters "y" and "u" in this poem.
   (c) Construct a diagram showing the relative frequencies of all vowels appearing in this excerpt.
2. A television station has conducted a poll among young people in order to determine typical viewing times. A total of 1000 people participated in the survey. The correlation between time of day and number of viewers is shown in the histogram (Fig. 1).
   (a) At what times does the number of viewers exceed 500? The total period of time when viewership exceeds 500 makes up what percentage of the total broadcast time?
   (b) How many people on average watch television for over an hour between the hours of 4 pm and 7 pm? What percentage of the total number of participants do they make up?
   (c) Determine the average number of viewers per hour.

The most important, most obvious, and sometimes the only possible means of solving a problem in probability and statistics is a computer. The following sample problems may serve to illustrate this point (Bunimovich and Bulychev, 2004):

1. Two people take turns tossing a coin: the first person to get "heads" wins. Evaluate the probability of victory for the first and the second player. To this end, conduct several experiments (as many as you think necessary), using (a) a table of random numbers, (b) a computer.
2. A pencil lead is arbitrarily broken into three pieces. What is the probability that these fragments will be able to form a triangle?

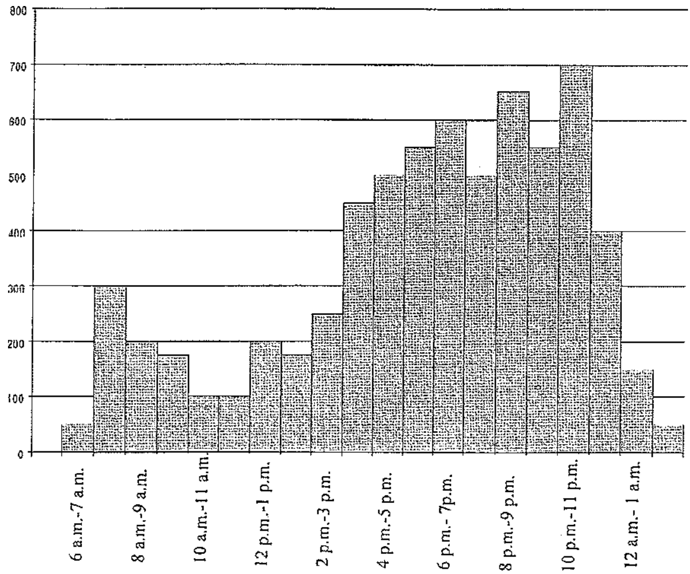

Fig. 1.

Find the answer through random modeling using (a) a table of random numbers, (b) a computer.

3. Eight passengers are riding in a bus that must make 10 stops. Each passenger has an equal chance of getting off at any one of the stops. Model a series of routes for such a bus using (a) a table of random numbers, (b) a computer. Use your model to determine the probability of the following events:

A = {all passengers get off at different stops};

B = {all passengers will get off at the same stop};

C = {somebody will get off at the fifth stop};

D = {nobody will get off at the fifth stop};

E = {somebody will get off at the first stop}.

5 First Results of Teaching the Experimental Curriculum

Because a stochastics curriculum is only now beginning to make its way into the general school, while in high schools it is still at the pilot stage, available only in certain regions, problems in probability and statistics have not yet appeared on the State Exams taken after the 9th year (basic school) or the 11th year (complete school).

Nevertheless, those regions that have introduced elements of combinatorics, probability, and statistics into the general curriculum do hold regional tests after each year of study. The content of these tests accurately reflects performance expectations at every level of study; thus, by looking at student performance, we can judge to what extent these expectations are being met. In conclusion, let us consider a test (Vysotsky and Borodkina, 2009) for the seventh grade, given in Moscow schools in 2009, along with some statistics on student performance.

Students have 45 min to complete their work. All necessary calculations may be carried out without a calculator; however, the students are permitted to use calculators.

Grading criteria:
The highest mark ("excellent") is given to students successfully completing four problems of their choice; the mark "good" is given to students successfully completing three of the problems below (a calculation error should not be penalized when it is evident that the student's reasoning is correct); "satisfactory" is given to students successfully completing two of the problems below, with a possible calculation error.

Problems:

1. The following table shows the duration of different vacation periods throughout the school year.

Fall	Winter	Spring	Summer	Days (total)
4	22	7	87	120

Which of the pie charts below accurately represents the distribution of vacation days as given in the table?

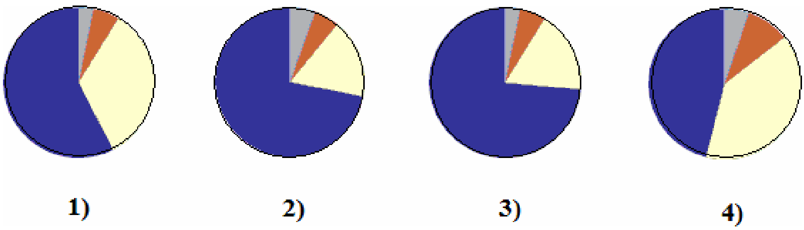

Fig. 2.

2. The diagram below gives the total number of factory workers in the Russian Federation in 1927 (numbers represent thousands of workers). Use the diagram to answer the following questions:

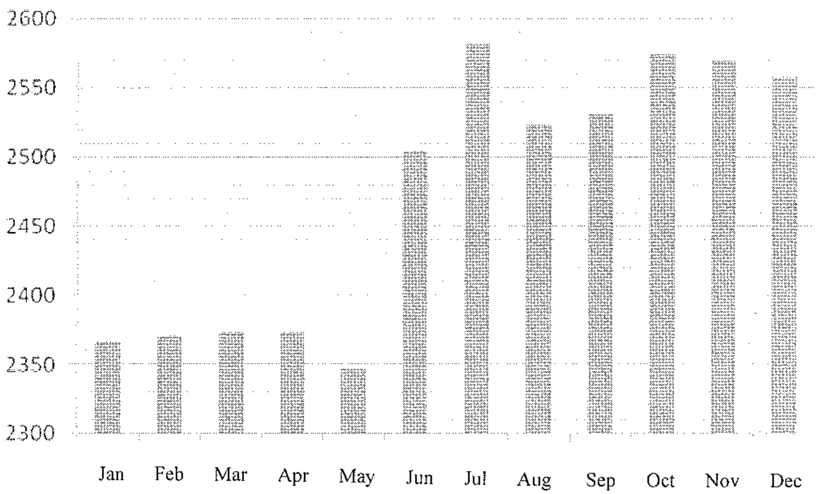

Fig. 3.

(a) Which month saw the sharpest rise in the labor force?

(b) Compare the number of factory workers in July with that in May. Give the approximate difference (in thousands of workers).

(c) Which months in the latter half of the year saw a drop in the number of workers?

3. The table below gives the number of Internet users in the 10 countries with the largest land areas in the world.

Country	Number of users (mln)
Russia	30
Canada	24
USA	220
China	213
Brazil	68
Australia	15
India	81
Argentina	11
Kazakhstan	2
Sudan	4

   (a) Find the arithmetic mean of the total number of users.
   (b) Find the median of the total number of users.
   (c) Which of the two values better represents the number of Internet users in these countries? Briefly explain your logic.

4. Swiss watchmakers use a special procedure to test the accuracy of their watches. The test measures errors in time-keeping (in seconds per 24 h period) at different temperatures, humidity levels, and positions of the mechanism. A watch receives a certificate of accuracy if the range of error does not exceed 4.5 s per 24 h period, with a dispersion less than 3 s. If the mean error in either direction exceeds 2 s, the watch must be recalibrated.

   The following table gives the results of five tests of the same mechanism.

Test number	1	2	3	4	5
Error(s)	−1.1	−2.7	−0.8	−5.5	−2.9

   (a) Find the mean error, range, and dispersion of error.
   (b) Determine whether this watch will receive a certificate of accuracy.
   (c) Determine whether the watch must be recalibrated.

5. The mean value of a set of numbers is 4; dispersion equals 18. Each number in this set was replaced by its opposite. Find:
   (a) the mean value of the new set;
   (b) the dispersion of the new set.

We can gauge the success of teaching this material — new for students and teachers alike — by performance statistics (Vysotsky, Borodkina, 2009, p. 50):

Number of classes taking the exam: 2,538
Number of schools administering the exam: 1,193
Number of students taking the exam: 52,900

	Grades			
	5 (excellent)	4 (good)	3 (satisfactory)	2 (poor)
Number of students	10,239	19,805	20,316	2,540
Percentage of students	19%	37%	38%	5%

The same data represented in a diagram:

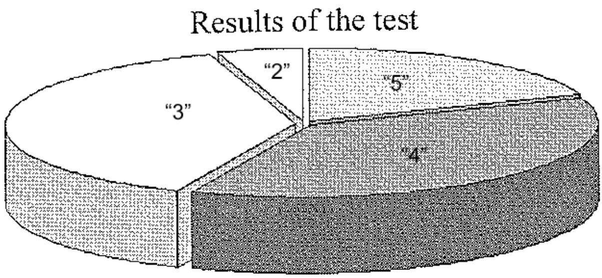

Fig. 4.

The following table shows to what extent each problem was solved.

	Problem No.											
Problem solved	1 (%)	2a (%)	2b (%)	2c (%)	3a (%)	3b (%)	3c (%)	4a (%)	4b (%)	4c (%)	5a (%)	5b (%)
Fully solved	82	75	76	67	90	83	68	40	45	47	29	24
Solved with minor deficiency	1	2	4	10	3	3	5	12	3	3	1	1
Partly solved	1	2	3	6	1	2	4	12	3	3	1	1
Incorrectly solved	13	20	15	13	5	9	14	18	17	14	7	9
Not attempted	3	1	2	3	2	3	10	17	31	34	60	64

The same numbers represented in a diagram:

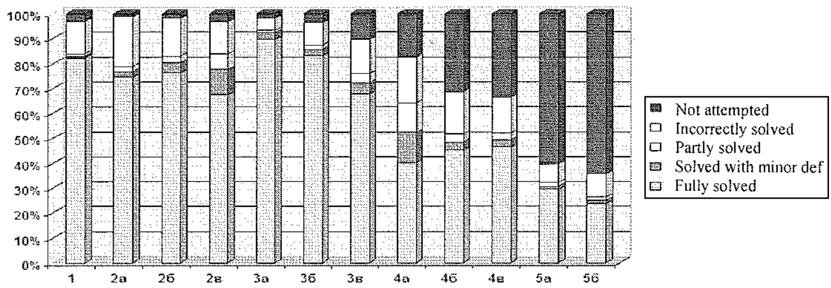

Fig. 5.

The following table and diagram show the breakdown by test score of the total number of participating students.

	Breakdown by score				
Score	[0:1]	(1:2]	(2:3]	(3:4]	(4:5]
% solved	(0–20)	(20–40)	(40–60)	(60–80)	(80–100)
Number of students	1067	2821	16,013	18,794	14,205
% of students	2%	5%	30%	36%	27%

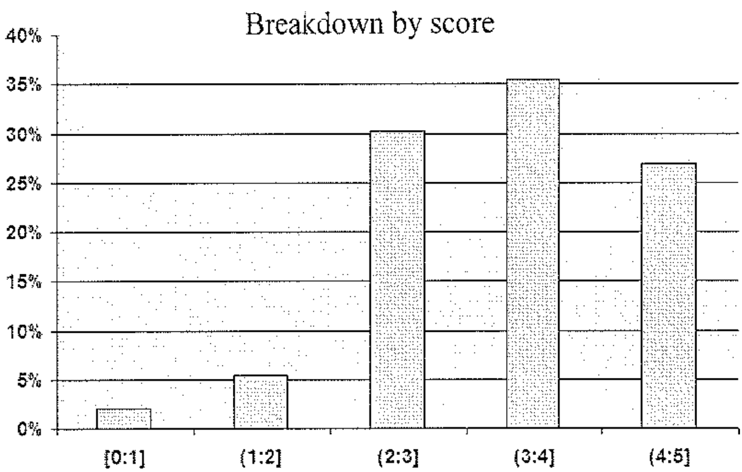

Fig. 6.

6 Conclusion

The foregoing discussion demonstrated that the integration of finite mathematics into the general secondary school curriculum is well underway in Russia. The era of initiatives, limited trials, and experiments has given way to a new era of finalizing standards and publishing textbooks and study guides — an era where finite mathematics is becoming a standard part of the general curriculum. At the same time, the scope of the material is continually widened: the recently approved new Federal Standard for elementary education includes for the first time an essential point among subject-specific "performance expectations" for elementary "Mathematics and Informatics:" "familiarity with the foundations of visual representation of data" and "ability to work with tables, charts, graphs, diagrams, sequences, sets; ability to represent, analyze, and interpret data" (Ministry, 2009, p. 12).

New Standards for the curriculum for basic schools are currently in preparation, and we can expect them to include an expanded section on finite mathematics. Appropriate teacher training is already underway in a number of school districts. For the first time in the history of the Russian school, combinatorics, probability, and statistics have a real chance of attaining the status of a full-fledged course of study.

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7

Schools with an Advanced Course in Mathematics and Schools with an Advanced Course in the Humanities

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1 Introduction

Currently, one of the most widespread expressions in the Russian pedagogical press is "profile" (or "profile classes"). It is assumed that virtually all students in the upper grades will choose special areas of focus — "profiles" — and consequently that all education, including mathematics education, will be constructed in accordance with these selected profiles. How this will be realized in practice, however, is not very clear, and the proposals that have been voiced do not seem promising to everyone [see, for example, the article by Bashmakov (2010a) in the first volume of this work]. There may even be reasons to fear that the ultimate outcome of these proposals will simply be a reduction in the scope of education and that, in the future, students in profile classes will be taught more or less the same material they were taught in ordinary classes in the past. On the other hand, it must be noted that the very notion that children in higher grades may have different interests, and that consequently it is necessary to acknowledge these differences, is universally recognized today.

This, however, has not always been the case. During the 1940s and 1950s, all schools had absolutely identical curricula. The prominent St. Petersburg teacher A. R. Maizelis related to the author of this chapter that, for example, an attempt made by one Leningrad teacher to continue using an older and more difficult edition of Larichev's (1952) problem book in her classes was severely punished by the educational administration, despite her good results. The more difficult was not permitted, nor was the more easy. However, the situation began to change by the very end of the 1950s, when classes and schools with an advanced course of study in mathematics began to appear. Almost all of today's Russian mathematicians passed through such schools. Graduates of these schools can be found among the mathematics faculty of any prestigious European or American university. The history of these schools, however, was dramatic and reflected the political and social processes that occurred in the country. Gorbachev's *perestroika* in the second half of the 1980s bestowed official praise and recognition on these schools, which, unfortunately, did not mean that their position improved. Almost at the same time, so-called humanities-oriented classes started being formed, in which an abridged course in mathematics was taught. Their history, although shorter, has also been complicated.

This chapter will focus on the history of mathematics education in schools and classes with an advanced course of study in mathematics on the one hand, and an advanced course of study in the humanities on the other. We will have occasion to address both social and purely methodological and pedagogical developments. Inevitably, certain details or facets will remain unexamined. In particular, we will limit ourselves to an overview of curricula, without attempting to shed light on all of their actual diversity; for example, we will not really enter into the subtle differences between the various approaches to presenting "advanced" mathematics in, say, the economic or natural scientific classes that appeared during the 1990s. Nor will we discuss in any detail the problems and assignments that are used in the schools described below, although these details are of great interest in our view. We also will not describe all schools that merit attention: we will be able — and even then only cursorily — to deal with the distinctive features of a small number of schools in Moscow and St. Petersburg (Leningrad).

2 The Appearance of Schools and Classes with an Advanced Course in Mathematics

During the second half of the 1950s, a broad campaign unfolded in Soviet (Russian) schools, calling for the "polytechnization of education" and the combination of education with "productive labor." A theoretical foundation for polytechnization was discovered in Marx and Engels, about whom the author of a modern textbook remarks, not without ironic condescension, that "they continued to adhere to utopian socialist ideas about the comprehensive development of the personality" (Dzhurinsky, 2004, p. 218). Indeed, the writings of Marx and Engels, as well as Lenin, contain pronouncements to the effect that, after spending four hours studying science, it is beneficial to spend four more hours engaged in physical labor, which is both good for one's health and conducive to the convergence of mental and physical labor, which was supposed to occur under communism (Bereday, Brickman, and Read, 1960; Lenin, 1980; Marx and Engels, 1978). After the Revolution, pedagogy set itself the specific task of creating a "labor school" (Blonsky, 1919), although the overwhelming majarity of the innovations introduced at this time were later declared to be "left-leaning perversions" (Karp, 2010a). The partial return to the previous point of view that took place after Stalin can be explained, of course, as arising from a desire to purify communist theory, but in our view it was more likely due to economic and political circumstances — for example, the shortage of workers in factories and collective farms.

Without discussing in detail the way that the struggle for polytechnization unfolded during the second half of the 1950s and the early 1960s, let us note that schools switched from a 10-year program to an 11-year program, with a large amount of time devoted to practical training during the education. Practical training could vary, however. Gugnin and Kirshner (1959) described how, from 1957 on, students from experimental classes at their school worked at an electronics factory. Initially, students worked three days per week (and attended school for three more), and at the factory they worked in 10 different shops. By the next year, the number of shops had shrunk considerably, and curricula and teaching methods had to be changed considerably. Gugnin and Kirshner also expressed a number of doubts concerning the organization of practical training, remarking that it would be ideal if students worked in the same educational shop under the supervision of methodologically competent supervisors.

Shortly after Gugnin and Kirshner's article was published, the number of different occupations for which students in this school (No. 38 in Leningrad) were being prepared decreased even further. The school began concentrating on preparing laboratory physicists, which led to greater attention to courses in physics and mathematics, student selection, and other issues. Gradually, the school came to be called a school with an advanced course of study in physics.

What happened with this school was not exceptional. A Leningrad mathematics teacher, who was among the first teachers to work in schools with an advanced course in mathematics (which started appearing around that time), made the following remark in an interview that the author of this chapter conducted:[1]

> Overall, this was the official situation: there were 11 grades then, and after completing 11 grades, children would receive a diploma showing that they had acquired some specialty. Our graduates were the first to receive diplomas that qualified them to work as computer programmers, and if they did not go on to college, they could go to work as programmers in the new computing centers that were being formed. Mathematicians immediately latched on to this situation and understood that it presented an absolutely fantastic opportunity to introduce serious mathematics education into the school. (Ryzhik, 2005)

The first classes with an advanced course in mathematics in the country began operating in September 1959 in school No. 425 in Moscow under the supervision of S. I. Shvartsburd. In the following passage, Shvartsburd (1963) made full use of the official terminology:

> The problem of preparing specialists mathematicians with a secondary education is becoming an important national economic problem. An especially large role in its solution can and must be played by mass secondary general-educational polytechnic labor schools with practical training. (p. 4)

[1]This and subsequent translations from Russian are by the author.

In practice, however, the schools with an advanced course of study in mathematics that began to appear — and starting with the following school year, 1960–61, their number grew quickly — turned out not to be especially mass-educational, but quite selective. Most of their graduates did not limit themselves to a secondary education, but continued their education in universities. In these classes, students really did have to work very hard: for example, in ninth grade, 11–12 hours per week were allocated for mathematics, which was twice the usual amount (Shvartsburd, 1963, pp. 138–146); furthermore, the course content was far more intensive and challenging. But this was, obviously, not exactly the labor that the propagandists of "productive labor" originally had in mind. As Shvartsburd (1963) very carefully wrote:

> After obtaining permission from and being well received by the Computational Center of the Academy of Sciences of the USSR, we set ourselves the goal of organizing practical training in such a way as to take up as little as possible of its employees' time and to use it as productively as possible. (p. 9)

In other words, direct work even at the Computational Center (let alone at a factory) was not supposed to take up too much time. Other schools that appeared after Shvartsburd's school operated according to the same schema. By July 1961, the education ministry of the RSFSR approved the first version of the basic documentation (curriculum, programs in the general course in mathematics and special academic subjects, etc.) for schools preparing computer programmers (Shvartsburd, 1963).

Thus, the original idea developed in a way that might appear paradoxical: the government had seemingly planned to force all students to engage in physical labor, and instead schools appeared in which students engaged in academic labor far more than they had done previously. The role of the schools' organizers — both the administrators, who were usually experienced in conducting business in the Soviet Union, and the mathematicians, who supported them — was quite great (thus, Shvartsburd cited the assistance received from the well-known mathematician N. Ya. Vilenkin, not to mention the then first deputy minister of education of the RSFSR, the well-known mathematician A. I. Markushevich; on the whole, schools with an advanced course of study in mathematics at once received the support of a wide circle of scientists). Yet it would be misguided to assume that they had managed to outwit the government, as if the government did not realize what was going on.

Of course, government leaders were to some extent constrained by the need to adhere to ideological dogmas; indeed, it may be supposed that they actually thought in these terms. But these individuals could not have been mistaken for sincere fighters for egalitarian communist ideals. For them, political reality was far more important.

One element in this reality was the need to engage in military-technological rivalry with the United States. This need spurred them variously to support (seek out, develop) the creators of the "nuclear shield of the homeland" — highly qualified scientists and engineers. Schools with an advanced course of study in mathematics were seen as a forge for such professionals. For example, in an interview with us, the well-known Moscow mathematics educator Vladimir Dubrovsky (2005), who worked at the famous Kolmogorov boarding school for the mathematically gifted, deliberately emphasized the role of the physicist Kikoin (who later became editor-in-chief of the magazine *Kvant*) in the creation of physics–mathematics boarding schools which will be discussed below: "He was more involved with government circles. After all, he was the third-ranking person in the atom bomb project."

The then first secretary of the communist party (CPSU) and the leader of the country, Khrushchev, had no intention of stopping at the 11-year school with polytechnic education. Today, formerly strictly classified materials from the Politburo (Presidium) of the Central Committee of the CPSU have become available, making it possible to judge what exactly the government was planning. A short note from December 23, 1963, reads: "All schools must be switched to a system of eight-year education. Talent selection: mathematicians, physicists, biologists, chemists" (Fursenko, 2004, p. 782). A more elaborate transcript preserves the argument behind this note. Khrushchev said:

> Some people say that in our age, the age of the atom and outer space, we need people with secondary education, we need mathematicians

> and others. This is a delusion, this is wrong. What is most important — precisely in our age, the age of cybernetics, automation, computing machines — is not theory, but practice. (p. 803)

In saying this, Khrushchev cited the prominent mathematician M. A. Lavrentiev, who had been invited to the meeting, remarking that school education was overloaded with useless information.

> Students spend 11 years sitting in schools and still come out as idiots, because if you're born that way, school won't give you more brains. And I agree with Comrade Lavrentiev: talents are born, one really has to be born a mathematician. (p. 804)

Khrushchev went on:

> Therefore, I believe that there must be a selection of mathematicians and that they must be educated from childhood. (p. 804)

Khrushchev saw no difficulty with stimulating and identifying talent: "If their genius hasn't blossomed now, it'll blossom when they're dying" (p. 804). Consequently, the education of the talented was conceived against the background of a reduction in general education: for the untalented, an eight-year school would suffice. In Khrushchev's speeches, one can detect inner doubts about the value of education, and even when he points out that not everyone can be sent to work in factories and uses Lenin as an example ["Take Lenin. What are you going to do — send Lenin to work in a factory, too? Lenin, a genius, who is born once in a century? That's not right" (p. 814)], he still cannot refrain from remarking: ". . . and yet I think that even Lenin, if he had not graduated from a gymnasium but had gone to work in a factory — he would have still been Lenin" (p. 814). Nonetheless, Khrushchev was able to suppress these feelings and support special education for the gifted.

Clearly, Khrushchev also had other considerations. If Stalin systematically shook up the party elite, making those who resided in party palaces one day move into prisons the next, and making their children leave Moscow's top schools for special orphanages for children of enemies of the people, then Khrushchev by and large abandoned such practices. This did not mean, however, that he was not frightened by the formation and development of a new class of the Soviet nomenklatura, and by the fact that the Soviet bureaucracy largely replenished its ranks by taking in the children of Soviet bureaucrats. Yegor Gaidar (1997), who became Acting Prime Minister under Yeltsin, and who had himself previously belonged to the Soviet nomenklatura by birth (even if not to its upper echelons), much later expressed the view that one of the reasons for the nomenklatura's dissatisfaction with the regime was that it was impossible to transfer positions by inheritance (pp. 120–121). Khrushchev recognized this desire for hereditary possession: "Let's take the lists of college graduates and see whose children they are" (p. 813). It turned out that the individuals who attended colleges, and who then entered the governing bureaucracy, were children of senior officials.

> Comrades, I think that among those of our children who received a higher education, at least 50% would not get into colleges. And I think that this would be a very good thing…there must be selection in life; he who wants to learn — he must show it with his persistence and labor…. (p. 814)

The transition to an eight-year education system was supposed to serve as a means for the creation of such selection. Under such circumstances, schools for the talented automatically became an alternative resource for replenishing the ranks of the country's upper classes (even if, possibly, not its uppermost class).

The model being created was clearly not without flaws. Khrushchev himself remarked that everyone tends to consider their children and grandchildren geniuses, and that it would be natural to fear that schools for the talented would become filled with the same children and grandchildren of senior officials. Experience showed, however, that this did not happen (at least, not then), possibly because by no means were all children of senior officials prepared to burden themselves with seriously studying mathematics. The strike that Khrushchev was planning obviously distressed the nomenklatura. At least, the transcript of a meeting of the Presidium of the Central Committee from October 13, 1964, during which Khrushchev was removed from power, opens with a list of questions for Khrushchev, the first of which is a question about eight-year schools (p. 862).

In 1963, physics–mathematics boarding schools appeared under the aegis of the leading universities. The first four opened in Moscow (probably the most famous of them is the Kolmogorov boarding school), Leningrad, Novosibirsk, and Kiev. In June 1964, the Ministry of Higher Education of the USSR passed a resolution concerning specialized boarding schools (Kolmogorov, Vavilov, and Tropin, 1981, p. 60). Subsequently, similar boarding schools (although with different characteristics) began to open in other Soviet cities with universities, first and foremost in capitals of republics.

The idea, which was supported by the leading mathematicians in the country, beginning with Andrey Kolmogorov and M. A. Lavrentiev, and picked up by broad sectors of the mathematical community, was to create opportunities for genuine and deep mathematics education for students from communities that were far removed from the Soviet Union's scientific centers. B. V. Gnedenko recalled that in numerous conversations with him:

> A. N. Kolmogorov repeatedly expressed the thought that very many mathematically talented students in villages and rural communities remain beyond the reach of the mathematics community, that it is impossible to organize mathematics circles and special groups for obtaining additional mathematical knowledge in all rural secondary schools, that it is impossible to supply such schools with qualified teachers who themselves participate in developing mathematical science. (Kolmogorov *et al.*, 1981, pp. 4–5)

The boarding schools were supposed to help solve this problem; in addition, they were supposed to help find and promote capable people from the provinces — people who would, incidentally, have no connections with the Moscow nomenklatura, since the newly created boarding schools were intended to refrain from accepting students from the cities in which they were located (this rule was sometimes slightly infringed, say, in Leningrad, but not, as far as can be judged, in Moscow).

The first period in the history of schools with an advanced course in mathematics was the most important; later, teachers who worked in these schools recalled this period as their glory days — it was then that the basic traditions were established, including the tradition of continuous interaction with research mathematicians (Sossinsky, 2010); it was then that the curricula and first didactic materials were created (all of this discussed below). It was then that specialized schools became known outside the country, exerting an influence on many other countries (Vogeli, 1968, 1997). A community of graduates from mathematics schools arose, which later played a very important role in the lives of these schools. "This was a territory of freedom," recalled the already-cited Vladimir Dubrovsky (2005), who was himself a graduate of Kolmogorov's boarding school. However, freedom, even highly limited freedom, soon came to an end.

3 Mathematics Schools During the Period of Stagnation and Later

Relatively soon after Khrushchev was overthrown, a period began in Soviet history that subsequently became known as the period of stagnation. A milestone was the invasion of Czechoslovakia in 1968. This event was followed by a quarter of a century — much of it spent under the rule of Leonid Brezhnev — during which all manifestations of liberalism, in both politics and economics, were increasingly stifled.

The position of the mathematics schools was contradictory. On the one hand, Brezhnev was engaged in the arms race and fighting for parity — as it was called — with the United States; to this end, it was necessary to prepare qualified workers. The party and Soviet nomenklatura, which during the Brezhnev years achieved the most comfortable position it had ever known, in principle favored special privileges for itself in virtually all fields; there were special stores for senior officials, special sanatoria, even special factory shops, and the like. In education, the role of such special institutions was played largely by schools with advanced courses in a foreign language; schools with an advanced course of study in mathematics invariably remained too difficult. Nonetheless, they, too, had sponsors among top government officials.

On the other hand, mathematics schools inevitably became hotbeds of independence, which top government officials found intolerable. Capable and confident students, who were above all encouraged to think, doubt, and ask questions, would begin to conduct themselves in the same way even in classes not devoted to physics or mathematics; this, naturally, could not be tolerated, even if it did not lead to any direct political actions (although the students of Leningrad's school No. 121 even distributed flyers). A teacher from a mathematics school who was interviewed by us related how he was regularly summoned by the Soviet political police (KGB) and questioned about the sentiments of the students (Karp, 2010b).

One may suppose that deeper feelings were also involved: the great 19th century Russian poet Nekrasov famously wrote about Lomonosov that "by his own will and by God's will, he became intelligent and great." The very possibility, assumed and encouraged in schools specializing in mathematics, of becoming great "by one's own will and by God's will" could not but provoke irritation within the rigidly organized system of the Soviet state.

Fields Medal winner Sergey Novikov (1996) wrote that "it is no secret that…the powers that be, often not without reason, found a spirit of dissent within the student population of special schools," which they attributed to "international imperialism and Zionism" (p. 34). Sossinsky (2010) describes how the fight against this malignant spirit was waged in practice. Elsewhere, he points out that during the 1970s "[the Kolmogorov boarding school] turned more and more into something like preparatory courses for students from the provinces, with the social background of the students playing a greater and greater role in their acceptance of the school, and their actual aptitude for science playing a lesser and lesser role" (Sossinsky, 1989). Ideally, the government wanted to continue obtaining the professional workers that it needed, but ones who would not — to use the colloquial expression — stick their noses where they did not belong.

This period has been described in other studies (Donoghue, Karp, and Vogeli, 2000; Karp, 2005). Here, we will confine ourselves to briefly analyzing one unpublished document, which can shed light on the official argumentation of the different sides as well as the situation as a whole.

In early 1974, the Minister of Education issued a special decree (No. 52), which indicated the strong and weak sides of schools with an advanced course in physics and mathematics (PhMSh). The decision was made to analyze the performance of such schools at the local level, and specifically in Leningrad, with a view of possibly shutting them down (just as school No. 121, mentioned above, was shut down). The party regional committee established a special commission. However, the very fact that it was headed not only by the director of the Institute of Teachers' Continuing Education but also by the well-known Leningrad geometrician V. A. Zalgaller, who had worked a great deal with students of specialized schools, indicated a favorable disposition by the regional committee. The commission investigated four Leningrad schools (Nos. 30, 139, 239, 470) and concluded that their work as a whole was successful. In particular, the following achievements were mentioned:

2. The best…PhMShs have eschewed the temptations of "parents' competition" and "narrow specialization." Their student bodies have good social compositions; they carry out instruction without weakening the nonprofile subjects….

   The PhMShs have become a significant part of the system for preparing specialists with a physics–mathematics profile…. In the physics and mathematics–mechanics departments of Leningrad State University, PhMSh graduates constitute one third of incoming classes, and during the years of study only 5% of them drop out, while 40%–50% of accepted graduates from other schools drop out during the years of study at the university.
3. The PhMShs have played and continue to play a crucial role in providing professional workers for the stock of computing machines….
4. The PhMShs improve the social composition of the community of specialists, opening a real path toward acquiring a specialty for children of working and peasant families. (Children from such families constitute 31% of students who were accepted at the university through the PhMShs, and 39% of students who were accepted through ordinary schools, but only 5% of those who went through the PhMShs drop out during the years of study at the university, while of those who went through ordinary schools, up to 60% drop out.)

5. PhMSh graduates…constitute the active core of the Komsomol, of student building groups; they begin engaging in scientific work earlier…they represent the majority of students who graduate with honors, of students accepted to graduate school. (Thus, in the graduating class of the mathematics–mechanics department in fall 1974, PhMSh graduates constituted 60% of students majoring in departments with an extended course of study, 64% of students graduating with honors, and 100% of students not from other cities who were accepted to graduate school). (LenGorONO, 1974, p. 59)

Shortcomings of the system were also noted. Most of them, however, were connected with the number of specialized mathematics classes (there were 56 graduating specialized classes in all, i.e. about 1500 graduating students), which was deemed excessive for Leningrad. In particular, it was pointed out that it would be more useful to organize entire schools with only specialized higher grades (such as Nos. 30, 38, 239, and 45, the boarding school at Leningrad State University), perhaps with the addition of eighth grades following the usual curriculum, rather than setting up separate specialized classes in ordinary schools. It was demonstrated that test results from schools Nos. 30 and 239 were significantly higher than test results from schools Nos. 139 and 470. The awards received by students in mathematics and physics Olympiads were counted for all classes. Awards for the leading schools are indicated in the table below; the other schools did not exceed three awards (p. 61).

Interestingly, the same report deliberately noted that it would be desirable for at least one deputy head of the city school board to be familiar with the curriculum in mathematics and science — a comment that revealed displeasure over the policies of city school board officials, who were obviously opposed to the specialized schools.

Table 1. Number of Olympiad awards by school.

School	45	30	239	121	38
Number	52	41	33	31	8

Despite this favorable report, even among the five best schools mentioned above, only two managed fully to survive: school No. 121 was soon shut down for political reasons, while school No. 38 was merged with school No. 30. Yet specialized schools survived this period and even made certain methodological–curricular advances, which will be discussed below.

Gorbachev's *perestroika*, which began in 1985, revived many hopes and processes that had characterized the Khrushchev years. Interest in specialized schools stopped being an exception. Newly fashionable slogans and goals that stressed "acceleration," "increasing productivity," and "attention to the human factor," aligned well with propaganda about the achievements of schools with an advanced course of study in physics and mathematics, while the limited freedom that belonged to the culture of such schools now had to be permitted in society in any case. CPSU Central Committee Secretary Yegor Ligachev (1988), speaking at a plenary meeting of the Central Committee, noted the achievements of Kolmogorov's boarding school and called for an expansion of the system of specialized mathematics schools.

As a result, the number of mathematics classes, for whose reduction the authors of the report cited above had made a case, began to grow rapidly. The education authorities were now quite favorably disposed to their proliferation, and more broadly, the rigid control of previous years became considerably weaker, not to say disappeared altogether (at the very least for economic reasons, although, of course, not only for them). Classes with an advanced course in mathematics were set up in practically every school (even if simply to prevent the "good children" from transferring to other schools).

During the Brezhnev years, a gap opened between the level of preparation in mathematics given in ordinary schools and the level of preparation that colleges required, and this gap was practically officially, or at least publicly, recognized. For example, the book of Chudovsky, Somova, and Zhokhov (1986) frankly states that it will examine problems "that are rarely encountered in classes in school," immediately explaining that these problems "appeared on college entrance exams" (p. 66). Consequently, many schools simply wanted to prepare students for entering college, without pursuing any more ambitious agenda. But if at one time six hours per week had been allocated for mathematics in ordinary schools, now a class would be dubbed a "mathematical class" and seven, eight, or sometimes even more hours would be allocated for mathematics. To see that studying mathematics "in depth" could have different meanings was not always easy.

The author of this chapter actively participated in the professional development of teachers for newly created specialized classes. The ramifications of this process were complicated. On the one hand, a relatively large number of teachers became acquainted with categories of problems and theoretical topics that were new to them, as well as with novel methods and methodologies for teaching; some of what had been created in preceding decades became accessible to and sought after by a comparatively wide range of teachers, and hence also a comparatively wide range of students. On the other hand, that which had been done with dozens of selected students could not be done with thousands. Naturally, not all teachers who taught the new classes had qualifications that could compare with the qualifications of the leading teachers from the old schools. Moreover, there were no such close interactions with research mathematicians in all of the newly created classes as had existed in the old schools — nor could there have been.

It should be noted that the processes occurring in society led to changes in specialized schools that were by no means always positive (although, of course, the greater openness brought new opportunities for those who worked in schools). The gradual opening up of the Soviet Union's (Russia's) borders resulted in great numbers of mathematicians leaving the country (completely or partly), and consequently their ties to schools (including schools from which they had themselves graduated and with which they had subsequently often actively collaborated) weakened. In general, the appearance of many new opportunities meant that working with strong students, which had previously been for many people practically the only acceptable form of public service, now became only one of many options. Individuals who had worked as teachers in mathematics schools were often those who, for various reasons, had been unable to find positions in universities or scientific research institutes (which, for example, were reluctant to hire Jews).

Now, places that had once been closed finally opened. More broadly, if in the past studying mathematics and physics, which as a rule were less ideology-laden than other fields, had attracted many students already for this reason alone, now new fields had become the most popular.

And yet the changes taking place in society made it possible during those years to open several new schools, which attracted and continue to attract strong students and which spearheaded new methodological approaches. During those years, new centers of mathematics education appeared, which were in great measure connected with specialized schools (most importantly, the Moscow Center of Continuing Education). Advances, first and foremost in methodological materials, including problems developed in schools with an advanced course of study in mathematics, became more accessible and widespread, if only because it became easier to publish (although the system of book distribution deteriorated considerably, so that books could often be found only by those who diligently sought them out, and even then, not always).

The most recent decade in the history of specialized schools is still too close for us to analyze objectively. Putin's Russia nominally unequivocally supports specialized schools; at least, it is easy to recall that when Russian president Dmitry Medvedev inaugurated the "year of education," he chose to visit one of the most famous physics–mathematics schools in the country (school No. 239). The future will reveal how the flourishing of specialized schools (even leading ones) will harmonize with reduced attention given to mathematics education in ordinary schools — a phenomenon that is much talked about, for example, in connection with the financial reform of the entire education system that has been enacted in recent years. Khrushchev, too, had the idea of giving a serious education in the higher grades only to talented students, but at that time there were far more resources (human, organizational, psychological, and even, apparently, financial) for education in lower grades, in which talent could manifest itself. Nor is it entirely clear how exactly talent will be identified.

Yet one thing is certain: Russian mathematics schools have already existed for more than a half-century. They have exerted a noticeable influence on education both within the country and abroad. They have produced important models for the organization of education and concrete methodological and instructional materials. It is to them that we will now turn.

4 The Everyday Life of Mathematics Schools

A substantial amount has been written about the life of mathematics schools. In addition to the already-mentioned publications, we would name Chubarikov and Pyryt (1993), Grigorenko and Clinkenbeart (1994), Karp (1992), Koval'dzhi (2006), and Tokar (1999). In what follows, we will inevitably skip over many details — to describe the life of many schools over a half-century is impossible — and concentrate on characteristics that may be said to be representative and in some sense idealized; we will focus on positive experience.

We must begin with the fact that mathematics schools were not much more expensive for the state than ordinary schools, if at all. In principle, these schools were financed according to the same schema as for all others. Admittedly, boarding schools virtually from the beginning of their existence had the option of paying for two teachers of some specialized classes (and Kolmogorov's boarding school, as far as is known to us, could even hire three teachers), which, in conjunction with their permission to organize classes more flexibly (for example, to conduct lectures for three or four classes at the same time), created certain opportunities for additional financing for certain clubs and circles, among other purposes. However, even in boarding schools, these sums were very modest; as for city schools, they operated mostly on a standard school budget, using funds for electives and mathematics circles, and subsequently also for other forms of individualized and consultation work that appeared much more recently. For a large part of the history of specialized schools, parents' resources were not solicited directly, although, of course, so-called sponsoring organizations were welcome in all Soviet schools, and the sponsoring organizations of mathematics schools were often more inclined to help out (for example, by donating their less than brand-new equipment to the schools). Even in recent years, as far as can be judged, the absolutely overwhelming majority of classes in mathematics schools have remained free for students. In general, even in cases where the budget of a mathematics school did turn out to be somewhat larger than the budget of an ordinary school, the significant differences had not so much concerned the school's equipment, let alone its facilities, as the human contribution to the school, which very frequently was made without compensation. Below, for example, we will describe a system that has evolved in Moscow's school No. 57, which involves the presence of five or six teachers in the same class. It is important to recognize that "extra" teachers receive no salary for such work or a salary that is purely symbolic (Davidovich, 2005).

Work in specialized schools was (and to a certain extent remains) prestigious. The opportunity to serve the mathematics community, the opportunity to work with an interesting group, the opportunity to interact with leading research mathematicians in school, the opportunity to make up one's own curriculum and implement one's own projects and not someone else's, the opportunity to return to the school from which one had graduated in a new capacity: all of these opportunities sustained the enthusiasm of those who had spent hours working with schoolchildren.

When specialized schools were first formed, they were made up of only the two highest grades, 9 and 10 (10 and 11 in the new system). In the 1980s, however, specialized classes for grades 7 and 8 (8 and 9) began to appear and rather quickly became common. Their proliferation was stimulated, on the one hand, by competition between specialized schools, which strove to attract the most capable students as early as possible, and, on the other hand, by the fall in the level of ordinary education — specialized schools preferred to prepare their own students for themselves, using the curriculum of the eight-year school (nine-year school). Note that the 1974 report quoted above already expressed the thought that specialized schools needed to include an eighth grade. Usually, specialized schools had several parallel tracks. For example, Kolmogorov *et al.* (1981) indicated that the Kolmogorov boarding school admitted 150 students for two-year schooling and 60 more students for one-year schooling (p. 11).

The work week in a specialized school is long. The number of hours allocated for mathematics and physics is considerably greater than that in ordinary schools. Standard Ministry of Education curricula from the end of the 20th century provide for the study in specialized schools of two mathematical subjects in grades 8–9 — algebra (5 hours per week) and geometry (3 hours per week); and in grades 10–11, they provide for the study of algebra and calculus (5–6 hours per week in grade 10 and 5 hours per week in grade 11, respectively) and geometry (3 hours per week) (Kuznetsova, 1998, p. 35). However, schools also had so-called elective hours, which at some specialized schools were made mandatory for all students; in addition, the standard class schedule included hours allocated for so-called *productive labor*, part of which usually went to programming and computational mathematics. The number of hours devoted to mathematics could thus reach 10, 11, or even 12 per week.

Other subjects were studied in accordance with the normal curriculum without any abridgments (for example, Kolmogorov *et al.*, 1981, p. 62). Moreover, although the number of hours allocated for other subjects was the same as in ordinary schools, not infrequently their actual requirements turned out to be higher, if for no other reason than simply that the students were on the whole stronger than usual.

Learning was not limited to ordinary classes, however. Extracurricular work was considered no less important. Schools usually offered many different clubs and electives (this time really not meant for all students). Their subject matter could be very diverse and could include quite advanced courses, which sometimes touched on unsolved problems [for example, the books of Alekseev (2001) and Zalgaller (1966) are based on the experience of such work with students]. Kolmogorov *et al.* (1981) mentioned such courses as “Finite Fields and Finite Geometries,” “Hyperbolic Geometry,” “Galois Theory,” “Elementary Mathematical Logic,” and “Elementary Number Theory” (p. 20). Such classes could also be devoted to various additional topics in school mathematics or, finally, to solving Olympiad problems.

Olympiad-related work occupies a very prominent place in specialized schools. For example, in school No. 30 in Leningrad (St. Petersburg), two “official” rounds of the school Olympiad were usually held every year. The first, a written round, was held instead of regular classes (three hours), and all students of the school participated in it. The winners were invited to take part in the second round, which, like the citywide round of the St. Petersburg Olympiad, was oral: the students explained their solutions to jury members, usually graduates of the same school (Karp, 1992). The level of problems in the second round usually approached that in the citywide round.

In addition, school No. 30 conducted annual tournaments of so-called "math battles" (Fomin *et al.*, 1996). Each class sent a team of seven students to such an event. To select the members for a team, a teacher (often with the help of graduates) would sometimes conduct an "unofficial" Olympiad within a class. Problem-solving contests, in both the written and the correspondence format, were also held at the school (Karp, 1992). Students from mathematics schools were also the most active participants in Olympiads outside the schools — in which, as has already been noted, they won the overwhelming majority of prizes.

Along with the "systematic" activities listed above, presentations by famous scientists, which periodically took place at the schools, played an important role. Andrey Kolmogorov gave regular presentations at the Moscow boarding school and even taught courses there. Other major mathematicians appeared in schools more rarely, but nonetheless it is clear that their lectures and their very presence were an important factor in the students' development. Not infrequently was it also possible to organize work for students under the direct supervision of research mathematicians on some research problem. Kolmogorov *et al.* (1981) noted that "once every two weeks a meeting of the Students' Scientific Society takes place, at which students report on their work" (p. 21). In other schools, school conferences were conducted; citywide and even All-Union (All-Russia) conferences were held as well, in which students from mathematics schools actively participated (Karp, 1992).

Extracurricular work was by no means limited to subjects related to physics and mathematics. Gnedenko recalled how Kolmogorov himself "lectured the students about the work of wonderful Russian and Soviet poets, about music, painting" (Kolmogorov *et al.*, 1981, p. 5). Lectures of this kind were read at mathematics schools, naturally, not only by mathematicians but also by representatives of the humanities; significantly, this was fully encouraged and promoted (at least as long as it did not meet with objections from the authorities, which, however, from the second half of the 1960s on, was by no means a rare occurrence — see, for example, Sossinsky, 2010). Literary evenings, collective readings of classics or modern authors, group field trips, and so on (see, for example, Karp, 2007) were all important components in the life of a mathematics school.

The enormous workload of students at mathematics schools meant that they had to be rigorously selected. Kolmogorov *et al.* (1981) related that admissions to the Kolmogorov boarding school were conducted in three rounds. The first round consisted of a written exam in mathematics and physics, administered in regional centers on the same days as the regional Olympiad (to save strong students from villages and small towns from extra travel). All students who could show a recommendation from their teachers would be allowed to take this exam. The second round was an oral exam for the winners of the written round. Based on the results of this round, some students would be invited to a selective summer camp (20 days), where, based on the results of their work in classes, final admissions would take place.

The selection of students for school No. 30 in St. Petersburg takes into account the results of Olympiads and contests, as well as recommendations by teachers of mathematics circles, and is made on the basis of "consultations" with the students (basically exams), which usually take place over several rounds — some written, some oral. It is important to hold several rounds in order to minimize the influence of accidents, reduce stress, and even acquaint students with the requirements; the ability to solve a problem better the second time around is considered an important indicator in the selection process (Karp, 1992). Moscow's school No. 57 selects its classes literally over a period of several years, observing the successes of students in Olympiads, inviting them to participate in mathematics circles, and conducting numerous consultations with them (Demidovich, 2005).

We have already noted that schools and classes with an advanced course of study in mathematics are not all identical. As an extreme case, particularly in recent decades, one can point to classes for which students are selected entirely from one ordinary school: the school administration and the teachers' council divide ninth graders into several tracks, based on their grades and, to some extent, on their wishes. Thus, for example, three classes might appear: a "mathematical" class, a "normal" class, and perhaps a "humanities-oriented" class (for which students might be selected on the basis of poor performance in mathematics, as will be discussed below). But in such "mathematical classes" the workload is usually considerably lighter.

In concluding this section, let us say a word about the teachers of mathematics schools (see also Karp, 2010b). When they first opened, mathematics schools needed remarkable people and attracted remarkable people. One example of such an unusual teacher was Anatoly Vaneev, whose higher education had been interrupted by World War II; after serving in the army, he spent a number of years in Stalin's labor camps. There, he came into contact with Lev Karsavin, one of Russia's major religious philosophers, and subsequently Vaneev himself became a notable religious thinker (Vaneev, 1990), which, not surprisingly, remained a secret from his students at school No. 30, and later from the teachers who attended his lectures at the Institute for the Continuing Education of Teachers. One of his school students, who subsequently became a well-known teacher at school No. 30 himself, was Vladimir Ilyin. As Ilyin (2005) recalled:

> Vaneev exerted a serious influence on me, although, of course, I found out about many things — the labor camps, the theology, etc. — only after graduating from school. But this, of course, could be felt in the breadth of his personality. I had a very good history teacher, Solomon Natanovich Ezersky. It was an absolute revelation to me that a history teacher could have other interests — Solomon Natanovich was a very active contributor to the magazine *Yunost'*, wrote novels, short stories. And what shocked me most of all was the fact that this could be discussed with students in class. This was one of the aspects of that special attitude that teachers had toward students, which had previously been completely unknown to me and which had a serious influence on me.

Other schools also had teachers of nonmathematical subjects who exerted a considerable influence on their students (see, for example, Sossinsky, 2010). Outstanding mathematics teachers came from different backgrounds. They included mathematicians — scientific workers, already mature or only starting out, who, coming to the school, were able to become wonderful teachers, finding ways to convey their understanding of and interest in mathematics to the children. They also included professional schoolteachers, who had previously worked in ordinary schools and who, coming to mathematics schools, were able to broaden their knowledge and horizons in a way that genuinely enabled them to teach their highly gifted students. Practically everyone who came to a mathematics school initially had to receive some additional education (in mathematics or practical pedagogy), but the very environment in the school — contacts and interactions with colleagues and research mathematicians and, most importantly, with strong students — facilitated the teachers' growth (Karp, 2010b).

In should be noted that during the period when specialized schools were being formed, their administrations were usually able to find and support remarkable people; and subsequently, too, a teacher who had educated a number of outstanding students (Olympiad winners, prominent young scientists, and so on) usually commanded a certain amount of respect, and hence enjoyed the administration's support. Specialized schools, which were based on selection, valued their reputations — that is to say, first and foremost, their teachers.

Naturally, there were limits here as well. The wonderful Leningrad teacher I. Ya. Verebeychik, because of whom school No. 121 achieved the Olympiad successes described above, was fired from the school during the aforementioned crackdown: the authorities determined that he was the least experienced teacher at the school, if only because he did not attend professional development courses (Verebeychik, 2005).

One can also point to cases in which, instead of being a community of people interested in mathematics and in science and culture in general, a school becomes simply a place where students can be decently prepared for college entrance exams, in an atmosphere that differs from the one described above. Yet, such developments are to some degree prevented by the intensive curriculum of the schools and the many long hours of work done together by students and teachers, which nurtures special relationships that last for years after the students graduate and which subsequently attracts graduates to return and help out in the schools.

5 Curricula, Textbooks, Approaches

5.1 *On Curricula*

The curricula of the first specialized schools were on the whole similar to the one described by Shvartsburd (1963). Mathematical subjects could be divided into special and general categories. Special subjects included "Computational Mathematics" (139 hours in all, over three years, grades 9–11), "Mathematical Machines and Programming" (156 hours), and practical work on computers (435 hours). General subjects were divided into "Algebra and Elementary Functions" (321 hours), "Calculus" (229 hours), and "Geometry" (270 hours) (p. 151).

It is easy to see that the number of hours allocated for mathematics was thus considerably greater than in ordinary schools. The hours allocated for the general subjects (and these are the subjects that in our view are the most important) were divided as follows:

Algebra and Elementary Functions

Grade 9

- Linear and quadratic functions, inequalities (15 hours)
- Powers with rational exponents (26 hours)
- Trigonometric functions of any angle (15 hours)
- Relations between trigonometric functions (13 hours)
- Reduction formulas and their corollaries (10 hours)
- Trigonometric addition theorems and their corollaries (25 hours)
- Exponential and logarithmic functions (36 hours)
- Review (11 hours)

Grade 10

- Linear algebra and elementary linear programming (50 hours)
- Complex numbers (12 hours)
- Polynomials and their properties (22 hours)
- Review (16 hours)

Grade 11

- Transcendental equations (18 hours)
- Combinatorics and elementary probability theory (22 hours)
- Review of the course "Algebra and Elementary Functions" and certain topics of the course in calculus (30 hours)

Calculus

Grade 9

- Measuring segments, real numbers (8 hours)
- Numerical sequences and limits (26 hours)
- The general concept of a function, the limit of a function (24 hours)
- The derivative and its applications (64 hours)
- Review (12 hours)

Grade 10

- The indefinite integral (20 hours)
- The definite integral (25 hours)
- Elementary differential equations (12 hours)
- Series (26 hours)
- Review (12 hours)

Geometry

Grade 9

- Vectors (14 hours)
- The coordinate method (40 hours)
- Metric relations in a triangle and solving triangles (20 hours)
- Geometric transformations (36 hours)
- Review (12 hours)

Grade 10

- Axioms of three-dimensional geometry and their corollaries (3 hours)
- Parallelism in space (14 hours)
- Perpendicularity in space (25 hours)
- The system of coordinates in space (12 hours)
- Polyhedra (24 hours)
- Review (6 hours)

Grade 11

- Solids of revolution (20 hours)
- Elementary mathematical logic, concluding remarks on the course in mathematics (20 hours)
- Review of plane and three-dimensional geometry, problem solving (30 hours)

We noted above that this curriculum *on the whole* conveys an idea of the curricula of the specialized schools when they first opened. This does not mean, however, that all the details were identical in every case, even in those years (and later on, changes were made to the numbers of hours and much else). Kolmogorov *et al.* (1981), for example, described the content of the geometry course taught at the Kolmogorov boarding school during the third and fourth semesters of a four-semester (two-year) course as follows:

> *Third semester.* Axioms of affine and projective planes and their models. Pascal's and Brianchon's theorems. Straightedge constructions. The Klein model of hyperbolic geometry.
>
> *Fourth semester.* Area and volume. Formulas for the volumes of the cylinder, the cone, the sphere and its parts. Simpson's formula. The Guldinus theorem. The area of a surface and the length of a curve. Oriented areas and volumes. The vector product and its uses. Measuring angles. Transformation of space. Euclidean space. (p. 17)

It is easy to see that this version of the course was more oriented toward university geometry than the former version, which to a very large degree coincided with what was taught in ordinary schools. We could give examples of cases in which topics usually studied in courses on abstract algebra were added to the program of schools with an advanced course of study in mathematics (Karp, 1992), and other examples will be given below. On the other hand, some of the topics listed above (such as linear algebra) are often not included in such courses. In general, as already noted, today, very different kinds of courses can lurk behind the label "advanced course," and naturally it is not possible for us to describe all of them. Instead, we will try to formulate certain principles, which may be considered common to all or almost all such courses.

In our view, this was done successfully already by Shvartsburd (1972). He wrote:

> Traditionally, the expression "advanced preparation in mathematics for students" in general educational schools has been understood to mean a heightened level of knowledge about elementary mathematics: a fluent and robust ability to carry out identity

> transformations, to solve equations and typical word problems, to compute the areas and volumes of figures…and so on. We give the notion of "advanced preparation in mathematics" a somewhat different pedagogical meaning. For us, it implies possessing certain knowledge and skills that lie beyond the bounds of the mandatory course, assimilating a number of new ideas and concepts, and grasping traditional topics in a more scientific fashion. (p. 17)

In other words, the hallmark of an advanced preparation in mathematics is not simply getting a high grade on a test that is given to everyone anyway, but knowing other topics as well, and perhaps most importantly, knowing them in a different manner. Shvartsburd (1972) went on to formulate the next (and, as he noted, the most important) principle: the need to establish close connections between the content of advanced preparation and the ordinary course in mathematics (p. 34). He underscored the fruitfulness of an approach in which "additional knowledge and skills are acquired by students in the context of a unified general course in mathematics" (p. 35). Such an approach naturally continues to stress fluency and robustness in the students' knowledge of the elementary course, but it also implies a fundamental enrichment of this knowledge, and not only as the result of an increase in the quantity of what is studied, but also as the result of new ideas introduced into the course in mathematics.

5.2 *On the Specifics of Teaching the Course in Mathematics*

Further discussion of the content of the course would probably not be comprehensible without a preliminary discussion on how the course was taught. Teachers whom we interviewed (Karp, 2010b) have stressed the importance of problem solving, through which practically all instruction was conducted ideally.

In a number of Moscow's schools (where the leading role was played by N. N. Konstantinov), a system of teaching had evolved already in the 1960s that was based on the independent solving by students of specially constructed sets of problems ("sheets"). In the introduction to their article, which may be described as a collection of problems, Gerver, Konstantinov, and Kushnerenko (1965) write:

> The problems presented here constitute a course in calculus. The collection contains the necessary definitions for independently solving all problems. By going over the material in this way, students master the techniques of mathematical thinking step by step. To master such techniques on a serious, professional level is the main aim of the course. (p. 41)

Obviously, a course constructed in this way implies a teaching process organized in a special manner. Davidovich, Pushkar', and Chekanov (2008), teachers at Moscow's school No. 57 who use this approach to teaching, preface their collection of "sheets" by explaining that five or six teachers must be present in the classroom at the same time. The "sheets" are handed out to the students (sometimes this is preceded by some brief explanation) and the students then solve them (at home or in class) and hand in their work to the teacher:

> The teacher can also discuss other ways of solving the same problems, go back to problems from older sheets that are connected with a new topic, formulate new definitions, and pose new problems (and receive their solutions from the students). One of the most important goals in all this is to fill in the "empty spaces" between problems, to create a holistic picture of the area being studied. (pp. 8–9)

Naturally, not all courses in all schools are structured in this manner. In the overwhelming majority of cases, lessons are outwardly quite traditional: there is one teacher who cannot listen to many responses simultaneously. Nonetheless, structuring a lesson as a system of problem-solving sessions, during the course of which students acquire the desired knowledge, is quite typical. R. Gordin, a teacher at the same school No. 57 who teaches geometry in the traditional manner (see, for example, Gordin, 2006), emphasized in an interview with us (2005) that problem solving usually arises in the course of class discussions, when students gradually improve and supplement one another's suggestions. The ability to structure a lesson in a corresponding manner, both in terms of selecting problems and in terms of organizing the discussion, is therefore quite important.

The most varied forms of working with problems are used: students are assigned problems for long-term periods and, conversely, they are given question-problems that require a quick response — make a prediction, formulate a hypothesis, or find a mistake; different solutions to the same problem are examined in class; oral and written problems are combined; and so on (Karp, 1992, 2010b). Once again, this does not mean that there can be no in-class lectures, explanations by teachers, or simply workshops during which students solve relatively routine (even if sufficiently technically difficult) exercises. All of this is also possible: a lecture that contextualizes what has been learned, analyzes what has been achieved, and poses new problems can sometimes be no less useful than the problem-solving sessions described above, nor can certain skills be formed without practice. There are also examples of an approach to teaching that outwardly resembles the traditional lecture–seminar system (Dynkin, 1967). What is important is that the spirit of research and the independent search for truth not be replaced by craftsmanship and the execution of commands and algorithms, however difficult they might be.

Below, we will discuss certain sections of the course taught in mathematics schools, including what would appear to be traditional college topics. It must be emphasized, therefore, that the "assimilation of new ideas and concepts," with which Shvartsburd connected the very notion of advanced preparation in mathematics in the passage quoted above, by no means implied "covering" college courses as quickly as possible: it was never anyone's goal to report cheerfully that students had already gone through, say, ordinary differential equations, or even partial differential equations, while they were still in school. The point was understood to be precisely the opposite: to examine what was being studied more attentively (and often for longer periods of time) than this was done in college. The aim was not only and even not mainly to learn a particular topic, but to develop "the techniques of mathematical thinking," as Gerver, Konstantinov, and Kushnerenko stated in the quote above. It is another matter that developing such techniques is impossible without a serious command of specific concrete mathematical material. What such material might consist of is the topic to which we will now turn.

5.3 *On the Content of Certain Topics in the Course*

The traditional, standard Russian school course in mathematics was (and remains) more proof-laden than, say, the American course. For example, in the course in geometry, practically all assertions were proven. Nonetheless, even this course was made into a more in-depth course not only by adding new sections but also by adding material to traditional sections. The most important items added, as already noted, were problems, and this was done in a way that often made it fundamentally impossible to divide the material into problems and theoretical content: a problem solved in class acquired the same rights as a theorem from the textbook.

Over time, the course in calculus became particularly important in schools with an advanced course of study in mathematics. Students who graduated from such schools usually went through a complete and proof-laden course in differential and integral calculus of one variable, which included the theory of limits and continuous functions. For example, the set of problems on the "Continuity of a Function" assigned to 10th-grade students in a four-year track at school No. 30 to solve on their own over a comparatively long period of time (2–3 weeks) included the following classic problems:

- Check the following function for continuity on the interval $(0, 1)$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q}. \end{cases}$$

- The function f is defined and continuous on the set of all real numbers. It is known that for any real numbers x and y, the equality $f(x + y) = f(x) + f(y)$ holds. Prove that there is such a number a that $f(x) = ax$ for any real x. (Karp, 1992, p. 76)

Quite often, the mathematical structure of the course followed that of college textbooks — for example, the classic textbook by Fikhtengolts (2001) — although, as we have noted, the pedagogical structure usually differed considerably from the college system of lecture–seminars. Sometimes, however, the course was structured completely differently. For example, Ionin (2005), a former teacher at the boarding school associated with St. Petersburg University, emphasized in an interview with us that it was considered very important at the boarding school to structure the course mathematically differently from the way it would be structured at the university, so that future students would not have to do the same thing a second time. Therefore, fundamentally new methodological–mathematical ideas arose. (For one example of a somewhat unusual presentation, see Kirillov, 1973.)

Consequently, the widespread alternative of acceleration vs. enrichment does not give an entirely accurate reflection of the possible choices of material for study. Certain topics were indeed drawn from what may be called college mathematics, i.e. one may indeed formally speak of acceleration, yet these courses were often structured differently even from a purely mathematical point of view. For one thing, they were structured in a way that placed greater emphasis, at least initially, on their connections with school mathematics. In addition, they were often designed to use a relatively small amount of material in order to present ideas which seemed important, but which in college courses often appeared later (for example, courses in calculus in specialized schools were often more "topological" than ordinary college courses). Among the topics that had been partly borrowed from the college program, calculus, as already noted, unquestionably occupied the most important position. But students were also taught abstract algebra, elementary number theory, the theory of polynomials, and certain topics from college geometry.

Topics that were usually not taught in colleges, however, turned out to be no less important. Sometimes, they came to specialized schools from mathematics circles; and sometimes one might say that they arose out of a careful exposition of the ordinary school course. For example, the ordinary Russian school course provides for the study of the concepts of the increase and decrease of a function or the range of a function (see Chapter 5 of this volume). Nonetheless, in nine-year schools, students usually "investigate" the properties of a function based on its graph, which in turn is constructed point by point, and there the matter ends. Meanwhile, much can be accomplished with a precise and deductive approach, long before derivatives are introduced, by relying on the completely proven properties of the quadratic trinomial and of inequalities. Such an elementary investigation of functions (in grades 8–9) allows students to concentrate precisely on the meaning of the concepts they are studying, rather than on the technique of using derivatives. The following problems may serve as examples:

- Find the range of the function $y = 3x - x^2$.
- Prove that the function $y = x^3 - 3x$ is increasing on the interval $[1, +\infty)$.
- Find the minimum of the function $y = \sqrt{4x^2 - 12x + 9} - 2$. (Galitsky, Goldman, and Zvavich, 1997, pp. 101–103)

To solve, for example, the first of these problems, it is enough to note that the range of the function is the totality of those y for which the quadratic equation $x^2 - 3x + y = 0$ is solvable; the range can, therefore, be found easily by writing the condition of the nonnegativity of the discriminant of this equation, $9 - 4y \geq 0$, from which we see that the range is the interval $\left(-\infty, 2\frac{1}{4}\right]$. Note that it is not sufficient to indicate that this function attains its maximum at $x = \frac{3}{2}$ (which is easy to determine by completing the square). It must be proven that the function attains all values that are less than the value of the function at $x = \frac{3}{2}$ (and students in grades 8–9 do not yet have the concept of continuity or limit). Discussing such topics helps students to understand more deeply what exactly is being proven, what exactly this or that concept consists of, what role definitions play, and so on.

The elementary investigation of functions can also touch on more complicated issues, such as convexity. Moreover, it may be connected with constructing graphs through geometric transformations. Once again, the ordinary school curriculum assumes that students will learn that, say, the graph of the function $y = x^2 + 1$ may be obtained from the graph of the function $y = x^2$ by means of a parallel translation upward along the y-axis by one unit. In classes with an advanced course of study in mathematics, students discuss far more intricate examples of both graphs and transformations (Karp, 1992). Note that simply constructing the graph of the function $y = \frac{1}{f(x)}$ by transforming the graph of the function $y = f(x)$, which is a relatively simple operation, opens up the possibility not only of appreciating the diversity of the transformations of the plane, but also of developing a "feel" for certain concepts (for example, the concept of the *infinitely large value*), which will subsequently be studied in courses in calculus.

Geometry offers many examples of such topics — topics that are not part of the college curriculum, but not entirely part of the ordinary school curriculum either. Lyapin (1967) described how students at a specialized school studied the geometry of transformations while solving construction problems [such courses were undoubtedly influenced by the books of Yaglom (1955, 1956)]. Other examples of topics studied in specialized schools include (Atanasyan *et al.*, 1996) inversion with respect to a circle, the classic theorems of elementary geometry (such as Simpson's or Euler's line theorems), and theorems about the collinearity of points and the concurrency of lines (Ceva, Menelaus, etc.).

The list of such topics, which lie, as it were, between ordinary schools and college, can be extended at length, but students at specialized schools also study a third category of topics that must be mentioned: traditional topics from the school course in mathematics. Their study of these topics differs from what goes on in ordinary schools — first, because it is more proof-laden and systematic, and second, because it includes more substantive and difficult problems.

It is clear, for example, that the presentation of the topic "Logarithmic and Exponential Functions" confronts the difficulty of defining a real power and of proving the continuity of the power, logarithmic, and exponential functions — a difficulty that is insurmountable in ordinary schools. Students in specialized schools possess a sufficient background to understand the essence of the problems that arise, and sufficient knowledge and techniques to overcome them with the teacher's guidance. It turns out, therefore, that the study of this topic in specialized schools unfolds in a completely different fashion from how this happens in ordinary schools. We will not discuss how this topic may be studied, however, but rather focus on the role that problems play.

In Russia, a tradition has evolved of writing and solving difficult problems on topics from the standard school course in mathematics. College entrance exams (traditionally conducted by each college separately), as well as graduation exams for specialized schools, have always been important sources of new problems, replenishing the stock of existing problems (currently, both college entrance exams and graduation exams have given way to the Uniform State Exam). Formally, these problems can be solved by any graduate of any ordinary school — in the sense that no special knowledge is required to solve them. Often, these problems can and even should be criticized for their artificiality and cumbersomeness (e.g. Bashmakov, 2010b). At the same time, not infrequently they contain substantive and beautiful ideas.

Admittedly, we are simplifying the situation somewhat when we speak about three sources of topics for schools with an advanced course in mathematics — traditional school topics, college mathematics, and topics "between the two" that are not typical of either schools or colleges. It is not always possible to make such precise distinctions, and in particular certain techniques for solving traditional school problems have effectively evolved into special topics themselves, which are studied in specialized schools and not in ordinary schools (this automatically places graduates of ordinary schools at a disadvantage on exams, notwithstanding any rhetoric that one or another problem may formally be solved by anyone).

Problems involving parameters have become an example of such a special topic or, more precisely, a running theme of the course in mathematics for specialized schools. Consider the following example of such a problem:

> For what values of the parameter a is there no value of x that simultaneously satisfies the inequalities $x^2 - ax < 0$ and $ax > 1$? (Galitsky *et al.*, 1997, p. 100)

The solution of the problem indeed does not require any special knowledge. It is sufficient to examine three cases. For $a > 0$, the solution to the first inequality is the interval $(0, a)$, while the solution to the second inequality is the interval $\left(\frac{1}{a}, +\infty\right)$. They do not intersect if $\frac{1}{a} \geq a$, which, given that $a > 0$, implies that $0 < a \leq 1$. Reasoning in an absolutely analogous fashion for $a < 0$, we obtain $-1 \leq a < 0$. It remains to be seen that $a = 0$ obviously works, since in this case each of the inequalities simply has no solutions. The final answer is $-1 \leq a \leq 1$.

Despite its technical simplicity, this problem is not so easy: it requires a certain use of logic and an ability to break down a problem into different cases and to examine them carefully. Naturally, experience in solving such problems helps students on exams and at the same time is beneficial to student development (again, if the concentration on this topic does not become excessive).

However, one can also give examples of many difficult and substantive problems that do not belong to a separate section. Such problems may be found in virtually any part of the school curriculum. Numerous problems also admit different solutions and solutions based on different parts of the course. Consider the following example:

> Determine the maximum of the expression $3x + 4y$, if $x^2 + y^2 = 25$.
> (For example, Zvavich *et al.*, 1994, p. 78)

Of course, this problem can be solved using differential calculus: it is sufficient to note that the maximum of the given expression is evidently attained when the values of x and y are nonnegative; then one can express, say, y in terms of x using the given equality, substitute it in the expression $3x + 4y$, and determine the maximum of the obtained expression with one variable using the standard algorithm.

The problem may be solved using far more elementary methods, however. One can, for example, see that since the expression

$$16x^2 - 24xy + 9y^2$$

is a perfect square and therefore nonnegative for all values of x and y, $(3x + 4y)^2 \leq 25x^2 + 25y^2$. From this, it immediately follows that $3x + 4y \leq 25$ (and the fact that equality is achieved is obvious, since it is achieved in the original inequality, $16x^2 - 24xy + 9y^2 \geq 0$).

Another solution may be obtained by writing the equality

$$3x + 4y = k \text{ (k is what is to be maximized),}$$

expressing, say, y in terms of k and x, and substituting it in the equality $x^2 + y^2 = 25$. It remains for one to find the greatest k for which the obtained quadratic equation has a solution.

An unexpected solution can be obtained using vectors. Indeed, consider the vectors (x, y) and $(3, 4)$. The expression $3x + 4y$ is obviously a scalar product of these vectors. But the scalar product of vectors, as we know, does not exceed the product of their lengths (but can be equal to this product). The length of one vector is $\sqrt{x^2 + y^2} = \sqrt{25} = 5$, and the length of the other is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$, which yields that the maximum is 25.

Interested readers may also find a purely geometric solution by investigating the behavior of secants and tangents to the circle $x^2 + y^2 = 25$, a trigonometric solution, and others as well.

To repeat, practice in solving difficult school problems is in our view extremely beneficial to the mathematical development of schoolchildren. It would be fair to say that all three of the approaches to selecting topics for study listed above are implemented in one form or another in every specialized school. It is another matter that the relation between them is by no means always identical. Today, in certain schools, already noted, teachers mainly give their students difficult school problems, sometimes forgetting that probably nothing can take the place of the experience of building a theory by constructing arguments and proofs (or building a theory by solving problems) and of working with difficult concepts not associated with school mathematics. Conversely, in some schools, students concentrate on mathematics that is not part of the ordinary curriculum, sometimes losing the connection with school mathematics and hurrying excessively, in our view, to move on to abstract and generalized concepts for which they are too young. The optimal relation between, and the optimal selection of, topics for study are determined first and foremost by the makeup of the student body. There are a variety of ways in which students can develop the "techniques of mathematical thinking" and be enriched by the experience of working with new, deeper ideas and concepts; what is important, however, is that educators set such objectives.

5.4 *On Textbooks for Schools with an Advanced Course of Study in Mathematics*

Highly selective schools can hardly expect to rely on textbooks published on a mass scale — the users of such textbooks would simply be too few in number. Up to a certain moment, schools with an advanced course of study in mathematics made do with their own handwritten materials (recall that even simple photocopying was very complicated in the Soviet Union). Along with materials that were dictated, copied by hand, or in exceptional cases photocopied, schools relied on college textbooks and problem books as well as handbooks for extracurricular work. The problem books and handbooks written at that time were sometimes published later, although not as texts for specialized schools (Bashmakov *et al.*, 2004; Sivashinsky, 1971), but as books and problem books for those interested in mathematics.

Special textbooks started appearing later, when the number of specialized schools increased. They included the textbooks of Vilenkin *et al.* (1972) and Vilenkin and Shvartsburd (1973). The latter, for example, was published in a comparatively large edition of 100,000 copies (textbooks for ordinary schools, however, were reissued every year in substantially greater numbers). Vilenkin and Shvartsburd (1973) included such chapters as:

- Real numbers
- Numerical sequences and limits
- Functions
- Derivatives
- Trigonometric functions
- Power, exponential, and logarithmic functions
- Elementary functions; transcendental equations and inequalities
- Integrals
- Series

In other words, the textbook included chapters from ordinary textbooks plus several special chapters.

It is difficult for us to judge how extensively these textbooks were used. We can be certain that for some schools these textbooks turned out to be too difficult and theoretical — for example, series were not taught in all schools, nor did students everywhere have such a sound grasp of, for example, the construction of the set of real numbers. For some schools, these textbooks were, on the contrary, not sufficiently proof-laden and deep. In addition, many schools adhered to the principle that teachers had to develop the theoretical part of the course on their own (naturally, relying on existing manuals), while problems would be gathered from different sources, including these textbooks.

The repeatedly reissued textbooks of Vilenkin, Ivashev-Musatov, and Shvartsburd (1995a, 1995b), even though their authors included the authors of the textbooks discussed above, were considerably different. In the first place, they were thinner: many topics and many assertions had disappeared (such as series). The textbooks came closer to the ordinary school curriculum. For good reason, a note in the first of them explicitly stated:

> The present volume is intended for a more thorough study of the 10th-grade course in mathematics in secondary schools, both for independent use and for use in classes at schools with a theoretically and practically advanced course in mathematics and its applications. (Vilenkin *et al.*, 1995a, p. 2)

One distinctive feature of this textbook was that it first discussed the limit of a function as the variable goes to infinity; then, as a special case, the limit of a sequence; and only then the limit of a function at a point.

Let us also mention the recently published textbook by Pratusevich, Stolbov, and Golovin (2009), written by teachers from St. Petersburg, including teachers from one of the oldest and most famous schools in the country — St. Petersburg's school No. 239. Its authors, however, describe its intended audience as follows:

> The textbook is intended for classes with an advanced level of mathematics education, in which no fewer than four hours per week are allocated for the study of algebra and elementary calculus. (p. 2)

In some mathematics schools, it should be noted, a substantially greater amount of time is allocated for the study of these subjects (for example, 6–7 hours). The textbook's authors explain that "certain sections have been deliberately left out (for example, the construction of a rigorous theory of real numbers), which mainly 'set the requisite rigorous tone' for the course in mathematics, but introduce no new tools for solving problems" (p. 408). The 10th-grade textbook contains the following chapters: Introduction (devoted to elementary logic and set theory); Integers (this chapter deals with congruences, prime numbers, and so on); Polynomials; Functions: Basic Concepts; Roots, Powers, Logarithms (notably, the authors confine their explanation of how one should understand irrational exponents by discussing an example, without offering a general definition); Trigonometry; and the Limit of a Sequence.

With regard to geometry textbooks, the first that must be mentioned is the textbook by Alexandrov, Werner, and Ryzhik (2006a, 2006b) for the upper grades of schools with an advanced course of study in mathematics, which was written in the early 1980s and has remained in use to this day. This textbook includes such chapters as "Transformations" or "Modern Geometry and the Theory of Relativity," while other chapters contain sections devoted to regular and semiregular polyhedra, spherical geometry, supporting planes, and other topics not studied in ordinary school.

Without attempting to characterize (or even mention) all of the currently existing textbooks (including textbooks for grades 8–9, which we are unable to discuss here, but which are published both as special books and as supplementary chapters to ordinary textbooks), we will just mention the relatively recently published textbook in geometry for higher grades by Potoskuev and Zvavich (2006, 2008), which is written from a somewhat different perspective than the textbook of Alexandrov *et al.*, and endows the course with additional depth while attempting to use simple and accessible language and approaches and, perhaps above all, using a thought-out system of difficult problems.

Generally, it seems to us that if writing a wide-audience textbook for classes with an advanced course of study in mathematics is an almost unsolvable problem because such schools are now simply too varied, then matters are easier with problem books since the teacher uses several books in any case and their diversity only helps the teacher. In addition to the problem books already cited, including Galitsky *et al.* (1997) and Zvavich *et al.* (1994), let us mention such problem books for upper grades as Galitsky, Moshkovich, and Shvartsburd (1986) or Karp (2006). By now, however, it is almost impossible to list all of the problem books that are currently in use.

6 Schools with a Humanities Orientation

While the history of schools with an advanced course of study in mathematics is already over 50 years old, the problem of creating a course in mathematics for schools with a humanities orientation has arisen relatively recently. Sarantsev (2003) even dated the origin of the problem exactly: "The problem of the humanitarization of education officially begins with the All-Union Congress of Public Education Workers (December 1988)" (p. 3). Without insisting on such an exact date, we must acknowledge that the resolutions passed by the Congress did indeed point out the need to rectify the unsatisfactory situation connected with the teaching of subjects in the humanities. On the crest of Gorbachev's *perestroika*, when it became commonplace to demand that "the human factor" be taken into account, and when it became fashionable to attack the older system for turning people into mere cogs in an enormous machine, Russian (Soviet) education began to be increasingly criticized for being excessively technocratic, with more and more voices demanding that it be "humanitarized."

What this term meant, however, remained sufficiently unclear. Chapter 10 of this volume discusses certain studies devoted to the humanitarization of education; here, we will merely refer to Sarantsev's (2003) overview, which lists a number of perspectives "on the content of the concept of the humanitarization of education in general, and mathematics education in particular." Among these different perspectives is an interpretation of humanitarization that equates it with increasing the number of hours allocated in school curricula for the study of subjects in the humanities; interpretations that emphasize the paramount importance of the developmental function of mathematics education; and interpretations that simply explain that humanitarization is a "complex, multifaceted phenomenon, characterized by a specific totality of characteristics" (p. 4).

Theoretical debates, however, have been accompanied by quite practical problems. We have already noted that school curricula became much more flexible in the early 1990s than they had been previously. While the number of hours allocated for each school subject had formerly been rigidly prescribed, now the Ministry of Education set only a certain minimum, and thereafter each school, within certain limits, was free either to increase it or to leave it at the minimum level. In the higher grades (10–11), a minimum of three hours was allocated for mathematics, and although, as already noted, some schools made use of their new freedom to increase the number of hours devoted to mathematics to eight or even ten, some schools also decided to give their students exactly three hours per week. These schools usually called themselves "humanities-oriented schools," and indeed the hours saved at the expense of mathematics (because the total number of hours was also fixed) were usually allocated for subjects that could be characterized as humanities.

Bearing in mind that students entered colleges by taking competitive entrance exams administered by each college individually, it is easy to understand that humanities-oriented students were sufficiently often defined (in practical terms, naturally, and not in rhetorical terms, for which much more elevated formulations were the norm) as ones who did not need to take entrance exams in mathematics. It is clear, however, that the group of students who did not take entrance exams in mathematics was very heterogeneous: it included both those who, for example, planned to enter the history department of a university and those who did not plan to obtain a higher education at all.

It must be pointed out that the Russian (Soviet) course in mathematics was indeed traditionally oriented toward preparing future engineers. The country offered all students the same course, which was, to a very great degree, aimed at the formation of firm skills in carrying out computations and technical transformations. Although the developmental role of mathematics — its role in teaching students how to reason, prove, and justify their conclusions — was always emphasized, it was still difficult to understand why exactly the ability to transform, say, trigonometric sums into products should occupy such a prominent place in the mental development of, say, a future singer.

Furthermore, while by the time mathematics schools were created very substantial experience in working with strong students (and particularly in working with mathematics circles) had already been accumulated on which educators could rely, nothing of the kind existed for working with those who did not plan to study mathematics. These students were taught in the same way as everyone else, except without achieving good results. Another distinguishing characteristic of mathematics schools in our view consisted in the fact that, while leading mathematicians participated in the creation of mathematics schools, the scientific community in the humanities (for objective reasons far weaker than the mathematics community) expressed no interest in participating in this way; consequently, it seems that the general level at which the problems of education were conceptualized and understood was far lower.

In this way, the problem of teaching mathematics to those who do not plan to study mathematics in the future — a meaningful problem that deserves attention — was posed under circumstances that were not particularly favorable. Let us name two more factors that made its solution difficult.

The first of these was the fact that reducing the technical skills that students acquired in the course in mathematics (a reduction that neither could nor should have been avoided) automatically deprived students in this course of the possibility of entering a technical college that administered an exam in mathematics (or, more precisely, made it impossible for them to enter such a college without additional study outside of school). Fifteen-year-old schoolchildren who had decided, together with their parents, that they would no longer have to take exams in mathematics because they were bound, say, for a career in law, would discover at 17 that, for one reason or another, they did in fact want to take an exam in mathematics, and a certain disappointment inevitably ensued.

The second factor was a certain apprehensiveness within the mathematics community about courses in mathematics for humanities-oriented students. This apprehensiveness stemmed from many causes — the transition to a new course always gives rise to apprehensions, because it requires new approaches of teachers. However, at that time, there were also quite well-founded fears that the entire traditional course would be eliminated under the banner of humanitarization; that the three-hour minimum would become the norm; and that extrapolating the notion that certain trigonometric formulas were useless for future scholars in the humanities would lead authorities to conclude that all of trigonometry was useless for everyone and that algebra and geometry were equally useless.

Notwithstanding these considerations, humanities-oriented schools did exist and a special course in mathematics was developed for them; some of these courses will be discussed below. While we have no statistics about the exact number of such schools, we can say that in St. Petersburg in the mid-1990s, graduates from such schools (classes) constituted approximately 5%–7% of all graduates from high schools. In the 1990s, special graduation exams in mathematics for such classes also appeared, which likewise indicated official recognition of the existence of this trend in education.

Subsequently, however, the space for such classes narrowed. This was connected in part with changes in the curriculum that made it much more natural to teach not one course (which came to be offered in humanities-oriented classes), but two traditional subjects — "Algebra and Elementary Calculus" and "Geometry." But the main reason lay in the transition to the Uniform State Exam, which is now offered to all students independently of the type of school that they attend, and which assumes a relatively high level of technical skill that is incompatible with what can be achieved in classes using the textbooks discussed below.

Thus, in our view, it may at present be said that while the history of mathematics in humanities-oriented schools has not ended, it has at least been interrupted. Not everyone will agree with this point of view, however, since among the variety of profile classes that are now coming into being, there are also classes oriented toward the humanities. Moreover, there already exist and will continue to appear textbooks and courses oriented to the ordinary basic course in mathematics, but stressing, for example, attention to history or art history, and therefore labeled as humanities-oriented.

Regardless of whether or not we will see a renewal of the teaching of mathematics in secondary schools in some format that is fundamentally different from the standard basic course (especially with regard to the technical skills that students are required to attain), the experience that we have had with such a form of mathematics education is important in itself. To some extent, it resembled what occurred in other countries, for example with the creation of so-called realistic mathematics (Gravemeijer, 1994), although it also contained many typically Russian attributes (Karp, 2000).

7 Curricula and Textbooks for Humanities-Oriented Schools

The following discussion will focus on three textbooks — Butuzov *et al.* (1995, 1996), Karp and Werner (2001, 2002), and Bashmakov (2004) — which appeared in the order indicated. Although unable to provide a detailed characterization of each of these courses here, we will nonetheless attempt to describe briefly what new elements, by comparison with standard, basic-level textbooks, were added to their content and what, on the contrary, was removed; in what way their style of presentation differed from that of the standard textbooks; and what aspects may be considered the most essential for the philosophy, as it were, of each of these courses.

The textbook of Butuzov *et al.* (1995) for 10th grade contains the following chapters: "First Acquaintance with the Personal Computer," "Numbers," "Functions," "Going into Space," "First Acquaintance with Probability," "Polyhedra," "Mathematics in Everyday Life," and "Different Problems." The 11th-grade textbook (Butuzov *et al.*, 1996) contains these chapters: "Dialogues About Statistics," "Objects and Surfaces of Rotation," "The Difference and the Differential, the Sum and the Integral," "How Volumes Are Measured and Computed," "Dr. Watson Becomes Acquainted with Combinatorics," "Symmetry," "Mathematics in Everyday Life," "The Horizons of Mathematics," and "Different Problems."

Clearly, many topics from the ordinary school course in mathematics are present here; however, the section on basic elementary functions (exponential, logarithmic, power, trigonometric) has been subjected to a radical abridgment or, more precisely, has been altogether eliminated, as has the section on equations and inequalities. On the other hand, the theory of probability has been added (which at that time was absent from the basic school course), and so have combinatorics and statistics; sections on mathematics in everyday life have appeared, as has a historical section. Complex numbers, which are missing from the ordinary school course, are mentioned, and in the historical chapter, for example, a whole section is devoted to Lobachevsky's geometry. At the same time, as the authors themselves note: "Explanations are often formulated only with the help of diagrams, figures, and visual representations; rigorous proofs are very rarely given" (Butuzov *et al.*, 1995, p. 3).

The style in which the textbooks were written is fundamentally different from the style of ordinary textbooks: one chapter is written in the form of a dialog; in another, the authors tell at length about the adventures of Baron Münchhausen; the title of a section in a third chapter — "We have company for dinner tonight: who's coming over tomorrow?" — would have naturally been impossible in a textbook for ordinary schools with its typically dry style.

The authors strove to be entertaining and, at the same time, "to convey an idea of the most fundamental mathematical concepts, knowledge of which … must be a part of the cultural background of a person in any profession." According to them, they likewise "attempted whenever possible to tell about the applications of mathematics in different areas of human activity" (p. 4).

The textbooks of Karp and Werner (2001, 2002) are in a certain sense more traditional. The 10th-grade textbook contains five chapters, of which the first, "Mathematics Around Us," is an introductory chapter which discusses the concept of the mathematical model and the notion of mathematical language, and also informally introduces the most important spatial figures. The second and third chapters ("Numbers and Counting" and "Functions and Transformations") in essence review, although at a higher level, the nine-year school course in mathematics. Then follow chapters on "Certain Elementary Functions" and "Elementary of Spatial Geometry," which contain traditional material (including elementary equations and inequalities), but greatly simplified from a technical point of view. The 11th-grade textbook has three chapters: "Elementary Calculus," "Elementary Computational Geometry," and "Introduction to Probability Theory and Mathematical Statistics."

Thus, although these textbooks do contain some material that is unusual for the ordinary school (including statistics, combinatorics, and probability theory, which was not usually studied in grades 10–11 at the time when the textbook was published, or the "mathematics of elections," which is briefly discussed in the 10th-grade textbook, or the whole discussion on the concept of mathematical modeling), on the whole the difference between this textbook and the textbooks used in ordinary classes is not in the material studied, but in the manner of its approach. Each chapter contains a relatively long, concluding section entitled "Read on Your Own," devoted to the history of mathematics: here, topics that are completely foreign to the standard curriculum, from abstract algebra to topology, are mentioned and briefly described; but this section is purely optional. In general, the material in the textbook is broken down into three levels: required material, which it is desirable for all students to learn; more difficult material, which is, however, offered to all students; and difficult material, which teachers might not even discuss in class, but simply offer for independent study. Published along with the textbooks were supplementary manuals, including problem books (Karp and Werner, 2002b; Karp, Werner, and Evstafieva, 2003). Both the textbooks and the problem books are written in a freer style than the one usually used in writing for ordinary schools, and the set of problems examined in them — including problems that draw on the humanities — is broader. For example, the textbook opens with a discussion on the concept of "rightness" in poetry and architecture, and the mathematical concepts underlying it. At the same time, the authors strove to write a book that would support the educational process as it has traditionally developed — with the formation of certain testable skills (never mind which skills), with tests, quizes and so on.

While the textbooks of Butuzov *et al.* (1995, 1996) or Karp and Werner (2001, 2002a) are intended for three hours of mathematics per week, the textbook by Bashmakov (2004) is intended for four or five hours per week, i.e. the same amount as in many classes that are considered ordinary. Nonetheless, this textbook is intended for use in one unified course and, above all, is structured in a fundamentally different way than the textbooks for ordinary schools; for this reason, we examine it here. It has seven chapters, but their titles do not always convey a full idea of their content. Along with Chapter 1, "Around Numbers"; Chapter 3, "Looking at Graphs"; and Chapter 4, "Learning Logic," it includes Chapter 5, "Moving Around a Circle" (devoted to trigonometry); and Chapter 6, "Who's Faster?" (about power, exponential, and logarithmic functions). Every chapter contains "Lessons" (which may take up several hours) and general "Conversations"; there are also "Entertaining Pages."

Below, for example, is the content of Chapter 7 ("Measure Twice"), which is intended to occupy 40 hours (with 4 hours per week allocated for mathematics), of which — in the author's view — 28 hours should be spent on "Lessons," 4 on "Conversations," and the rest on tests and research work (p. 317):

- Introductory Conversation
- Lesson No. 38: Area
- Lesson No. 39: Volume
- Conversation: Differentiation and Integration
- Lesson No. 40: The Integral and Area
- Lesson No. 41: Measuring Geometric Magnitudes
- Lesson No. 42: Finite Sets
- Lesson No. 43: Probability
- Lesson No. 44: Repeated Trials
- Conversation: Mathematical Expectation
- Entertaining Page: Great Ideas of Great Minds

The author notes:

> It would be wrong to imagine that the basic characteristic of this course is a reduction in the content of the ordinary school curriculum to the required minimum. On the contrary, it includes many concepts, facts, and even whole sections that are absent from the standard course (complex numbers, statistics, probability, quantifiers, interpolation, etc.). The most important changes are changes in emphasis. (p. 4)

Further, the author stresses that the textbook may be used in different ways: students can "limit themselves to superficial theoretical facts" or they can "approach a sufficiently high understanding of the material." In conclusion, the author emphasizes that his "book is less a finished and thoroughly tested course than a guidepost to be used in the important and difficult work of turning mathematics into a tool of cultural development, into a part of one's spiritual life" (p. 4). This textbook was accompanied by a separate problem book (Bashmakov, 2005).

There is probably no need for the author of this chapter to conceal the fact that he is, of course, most partial to the textbook of Karp and

Werner (2001, 2002a). The authors of this textbook strove to design a unified course in algebra and geometry on the basis of the mathematical modeling of real-world processes, combining the visual and the logical with an analysis of the sequential (historical) development of ideas (Karp, 2000). But no less important for them than *modeling*, *visual representation*, and *historicism* was a principle that one might like to call "*realism*."

The teachers' manual to this textbook states: "The knowledge and skills that students in classes with a humanities profile are required to possess are somewhat different from what is required of them in general educational classes, but this does not mean that there are no such requirements at all and that they are replaced with empty talk" (Karp and Evstafieva, 2003, p. 3). Since one of the principal goals is to teach students to reason mathematically and to work with mathematical concepts (including concepts that arise in the course of modeling), enough time must be set aside for such reasoning. This means that new concepts must be relatively few; at the same time, teachers must be prepared to teach students how to reason in this way, which in our view means that they must not stray too far from traditional subject matter.

Putting aside this analysis of specific textbooks, we should say that understanding how graphs and definitions may be used to solve elementary exponential or logarithmic equations and inequalities, and understanding how and why it is necessary to solve such equations and inequalities, are both forms of genuine mathematical activity, which is useful even to those who will never need to solve these equations again (by contrast, say, with memorizing various formulas and practicing applying them). Naturally, it would be desirable to publish the most-varied popular books, which would use an entertaining and accessible style to tell those who are interested — including those who have decided to go into the humanities — about some more unusual topics in mathematics as well. Moreover, it would be desirable for textbooks to contain some information (perhaps in optional sections) that would let students know that there are many curious things beyond what is studied by everyone in class, but what is studied by everyone in class under the teacher's supervision, in our view, must be limited and clearly structured.

The job of the author of a textbook is to ascertain that even in classes with a small number of hours, problem solving and mathematical reasoning are still genuinely present, and are not merely nominal requirements. Let us repeat that, in our view, such genuine mathematical activity can be set in motion even by using a comparatively small range of concepts — how to draw or glue a polyhedron; how to determine a grade for a semester based on all the grades given during the semester, but in such a way that the grade on the final exam has greater weight than the others; how to determine the least expensive way to ride the metro over some period of time, given a certain system of discounts; how to compare various given numbers; how to determine whether the graph of a function can possess this or that property; and so on. These and many other questions may serve as a foundation for authentic and substantive mathematical activity.

Of course, such activity must exist in ordinary classes as well. Yet, while its absence from ordinary classes is at least compensated for by the consolation that students may still have learned certain algorithms which they will need in the future, the challenge in classes with a humanities profile is more acute. Either we must try to achieve something mathematically substantive using accessible materials, or we must acknowledge that teaching mathematics is fruitless. This especially endowed the thinking behind the writing of the various textbooks for classes with a humanities profile with such importance.

8 Conclusion

This chapter has focused on "unusual" schools, but their existence, and the methodological and mathematical ideas that came out of working in them, have exerted an influence on all schools. "Unusual" schools have appeared during political transformations that occurred in the country, and in general their fate has become closely connected with the political climate in Russia: stagnation (also called "stability") turned out not to be beneficial to them, and this non-methodological side of their history also makes them interesting to study. Mathematics schools quickly became known abroad and were imitated. Incomparably less has been written about humanities-oriented schools and, indeed, if the successes of mathematics schools are obvious, then, with respect to the teaching of mathematics in humanities-oriented schools, it is not even very clear how success should be measured and a consensus about what constitutes such a success is unlikely. However, comparison of work done in this area, both in Russia and beyond its borders, is in our view also worthwhile.

The slogan of differentiation in mathematics education is popular now in many countries. What lies behind it is the recognition that people are different and that different sides of mathematics may be closer to them, and consequently, that they should be taught in different ways. In our view, the Russian (Soviet) experience, an important side of which is the teaching of mathematics in mathematics and humanities-oriented classes, deserves attention.

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8

Assessment in Mathematics in Russian Schools

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1 Introduction

The structure and organization of Soviet schools mandated the continual assessment of what was happening within them from top to bottom. "Monitoring and control" was a crucially important part of the work of the school administration and the education authorities at the district, municipal, republic, and national levels. Consequently, an enormous amount of attention was devoted to assessing the achievements of students in general and in mathematics in particular. Paradoxically, national statistics about what students actually knew turned out to be utterly unreliable, and everyone recognized this fact, including the top education authorities.

In 1989, one of the authors of this chapter was present at a meeting of the board of the USSR State Committee on Education and Science (which included the Ministry of Education), at which one of the Committee's top-ranking members remarked that it was still unknown how many Soviet schoolchildren had learned integrals. "One half of what the official statistics tell us? That would be an absolutely wonderful achievement. But we just do not know." The accumulation of all conceivable kinds of information, coupled with the constantly repeated demand that results be improved, led to falsifications on a massive scale. This fact deserves our attention today, when many regard recording a teacher's or student's every move on a computer as the principal means of improving education.

Meanwhile, on the level of individual classes, genuine assessment turned out to be indispensable and even, to a certain extent, inevitable. Consequently, even in these conditions, Soviet and Russian mathematics teachers and educators devised and applied many methods and techniques for assessment, which today come across as both interesting and useful. In this chapter, we will focus predominantly on describing specifically the practical side of assessments with which students come into contact, only dealing marginally with the analysis of mathematical assessment from other points of view. Nor do we always specify whether we are talking about formative or summative assessment — the same approach to organizing assessment, and the very same tasks, may be used both to assess formally the level a student reaches and to help students themselves assess their own knowledge — while enabling the teacher to obtain a rough and informal idea of the effectiveness of the teaching process.

There are numerous ways to classify different approaches to assessing and monitoring students' knowledge. For example, some researchers distinguish between *continuous* monitoring, which takes place as part of the teaching process; *thematic* monitoring, which takes place at the end of the study of a specific topic; and *final* monitoring, which takes place at the end of the semester, the year, or the entire schooling process (Stefanova and Podkhodova, 2005; Temerbekova, 2003). Another approach distinguishes between oral, written, and practical forms of assessment, oral survey tests, and exams (Temerbekova, 2003). Below, we mainly follow the latter approach.

2 General Assessment Issues

2.1 *What Is Assessed and Why?*

The introduction to a collection of articles edited by the well-known psychologist Yakimanskaya (1990) points out that assessments are conducted in schools "based mainly on the final outcomes of knowledge acquisition.... Knowledge is assessed mainly by evaluating acquisition at three levels: by asking students to reproduce knowledge that has been learned, to apply it based on a given model, and to use it in a new, nonstandard situation" (p. 3).[1] However, this taxonomy, which resembles that of Bloom (1956), seemed insufficient to the authors of the collection: "Such a system of criteria fails to take into account the psychological nature of knowledge acquisition, the process of knowledge formation; it leads to a rupture between the characteristics of the knowledge that students ultimately acquire and the process of its acquisition" (p. 3). Instead, the authors suggest, "assessment must reflect the process side of knowledge acquisition" (Shiryaeva, 1990; p. 93). Moreover, Shiryaeva explains that even when the subject being discussed is testing students' knowledge acquisition, it should be borne in mind that students acquire knowledge of a particular type; namely, she writes (quoting Yakimanskaya, 1985), "[knowledge] about the substance and sequence of mental actions (operations) whose implementation facilitates the acquisition of scientific knowledge about a domain-specific reality" (p. 17).

Constructing a system for the objective assessment of the individual process of learning mathematics — or a system for evaluating the formation of methods of scientific cognition — is an alluring but highly problematic task. Yet there exists a contrary and popular tendency to make assessment something far more concrete, for example by using such assessment tools as problem sets of minimal (so-called mandatory) and higher levels, with the provision that the "orientational contents of the minimal-level problems included in these sets ... must be open for all participants in the educational process" (Dorofeev *et al.*, 2000;

[1] This and subsequent translations from Russian are by Alexander Karp.

p. 41) — in other words, that students be informed prior to taking a test what kinds of minimal-level problems will appear on it. The critics of this approach believe that this would merely steer the vast majority of students toward rote learning, the pointlessness of which is augmented in their opinion by the fact that the value of isolated, highly specific skills in mathematics can, in view of the current technological development, be put in question.

The issues raised above are at the heart of current debates about reforming the system of exams, which will be discussed below. However, they are not new. A teacher ideally aims to take into account everything: the proficiency that a student ultimately achieves in solving concrete problems, the extent of the student's acquisition of general intellectual skills, and the manner in which the learning process unfolds.

Temerbekova (2003) identified the following functions of assessment:

- Monitoring and diagnostic functions;
- Educational functions (since students undergoing monitoring have a chance, for example, to systematize their knowledge);
- Stimulative functions (since assessment, generally speaking, facilitates the development of students' learning activity);
- Formative functions (since, once again generally speaking, monitoring facilitates the development of a sense of responsibility);
- Prognostic functions (monitoring may be used as a basis for arriving at some kind of prognosis about the development of a student's education, and thus also as a basis for taking corrective measures if necessary).

To these functions may be added that assessment also dictates the school curriculum, and although it does so in a condensed fashion, its impact is considerable. Teachers teach that which will later be tested. It is not difficult to recall, for example, that such a part of the school course in mathematics — as the elimination of extraneous solutions that lie outside the domain of the functions being examined — has assumed completely disproportionate importance in the teaching of mathematics, despite not being prescribed by any program (Boltyansky, 2009). This came about because problems associated with this idea often appeared on exams. Today, the format of problems on the

Uniform State Exam (USE) is the subject of heated debates precisely because new types of problems will exert an unavoidable influence on the school course in mathematics.

Without going into further details here, let us merely emphasize that assessment can also have a destructive influence, pushing students away from mathematics and accustoming them to irresponsibility and even dishonesty. This fact makes the task of the teacher in conducting assessments all the more difficult and important.

2.2 *Assessment in the Past*

Within the bounds of this relatively short chapter, it is impossible to provide anything close to a detailed analysis of how the outcomes of mathematics education were assessed in the past. We can simply note that the history of assessment in Russia — particularly the history of exams in mathematics — includes many dramatic transformations (Karp, 2007a).

Literature and journalism have preserved for us numerous images of czarist era teachers, who "pounce on everything in sight" when they administered exams (Averchenko, 1990). Indeed, it was officially expected that assessment would be conducted in an extremely strict and impartial fashion. The rules for testing students of gymnasia and pregymnasia published by the Ministry of Public Education (1891), for example, stipulated that "students taking exams are not allowed to use aids of any kind, including dictionaries, with the exception of tables of logarithms when taking exams in mathematics" (par. 13b). Furthermore, it was explained that "oral exams are conducted before a commission consisting of the director or inspector, or the class preceptor and two teachers" (Sbornik, 1895, par. 33). During written exams, every student had to sit at a separate desk and remain under the constant supervision of specially appointed overseers. Students with unsatisfactory grades in mathematics (or in Russian language, Latin, and Greek) were not admitted to the exams, and those who had failed an exam, which happened not infrequently, had to take it again in the fall. Students who failed to pass the exam a second time would often be held back for another year. There were even rare cases of students being held back for three years (although, generally speaking, usually in such cases students would simply get expelled from school). The following excerpt from the *Complete Collection of Documents of the Russian Empire* is representative of the centralized system of governance that had taken shape in the country:

> August 17 [1881]. Report by the Acting Vice-Minister of Public Education,…, most humbly submitted [to the czar]. Concerning the decision to hold back second graders Zilberstein and Monchinsky of the Chenstokhovsky Pregymnasium for a third year in the same grade.
>
> The Superintendent of the Warsaw school district has submitted for review to the Ministry of Public Education a petition from the Inspector of the Chenstokhovsky Pregymnasium concerning holding back Henrich Zilberstein and Anton Monchinsky, second graders at the Pregymnasium, for a third year in the same grade. In accordance with §34 of the Statute Concerning Gymnasia and Pregymnasia, students may remain no more than two years in the same grade. In view of the fact that the aforementioned students, distinguished by their exemplary conduct and unflagging diligence, were unable to pass to the next-highest grade due to frequent illnesses, I take upon myself the responsibility of most humbly requesting Your Imperial Majesty's most gracious consent to hold back Pregymnasium students Zilberstein and Monchinsky in the same grade for a third year. (PSR, 1893, #360a)

The same document also contains the following note: "The original indicates that it was 'ratified by the Supreme Authority.' The Sovereign, however, has indicated that in the future such decisions may also be made independently by the Superintendent of the school district."

Meanwhile, a great deal of evidence has survived about widespread cheating on exams and in the educational process in general (Karp, 2007a). Public opinion about such cheating, as far as can be judged, was favorable, since the gymnasium's insistence on high standards was seen as an expression of the dictatorial tendencies of the czarist regime. It is not surprising that after the Revolution of 1917, exams were abolished, along with most other traditional forms of individual assessment. The laboratory team method gained popularity, in which students worked on assignments in teams; during the final class, the work of each team would be assessed as a whole, without taking into account the students' individual contributions (Glebova, 2003).

During the 1930s, this system of assessment, based on projects and on what today would be called a "portfolio," was, along with other innovations of the 1920s, declared a leftist deviation (Karp, 2010), and the system in many respects reverted quite rapidly to its pre-Revolution form, with a renewed emphasis on strictness accompanied by frequent cheating and falsification. In addition, it became considerably more centralized. Prior to 1917, it was impossible even to imagine that, for example, the whole country from the Baltic Sea to the Pacific Ocean would take the exact same exam in mathematics, composed in Moscow, but precisely such a system became established by the mid-1940s. At the same time, falsifications became more and more prevalent: in Stalin's time, almost 20% of the students received failing grades, while during the Brezhnev years, this number had dropped to 1–2%. At a certain point, the idea of instruction without formal grades, particularly in elementary schools, became popular again, although the *Pedagogical Encyclopedic Dictionary* (Glebova, 2003) characterizes these ideas as being difficult to realize.

We repeat that it is impossible to undertake a detailed analysis of the history of assessment in Russia here. Nonetheless, it is important to recognize that traditions that have existed to this day took shape over decades, and that mathematics educators, influenced by these developments, formed specific beliefs about assessment and its role — beliefs that were by no means identical in all respects to the views of educators in other countries.

2.3 *Some Facts About the Organization of the Teaching Process and of Assessment*

Russian schools have always given grades on a scale from one to five. The top grade was five ("excellent"), followed by four ("good") and three ("satisfactory"). These three grades were considered positive. Grades two and one were unsatisfactory and effectively indistinguishable from each other. Thus, Russian schools have adhered and continue to adhere to what is effectively a four-grade scale. Periodic attempts to change it have usually been proposals to introduce a 100-point or a 10-point scale, or proposals that grade two be changed into a passing grade, but none of these proposals were ever realized. A four-grade scale places both teachers and students in a rather difficult position. Not every two fives are equal to each other, even if they are given by the same teacher. The same can be said of two fours and, even more so, two threes. About grade three in general, it has often been said that this is a "rubber" grade which is given to students both for work that is quite decent — although not on the level of a four — and whenever the teacher simply does not wish to give a two. Teachers have tried to get out of this predicament by adding pluses and minuses to the grades ("five minus," "four plus," "three minus," and so on), but this is prohibited by the Ministry of Education's instructions. In students' workbooks, however, one can come across such wonderful grades as "three minus minus."

Concrete, specific criteria for grading were developed for different subjects in the secondary school curriculum. They defined the requirements that a student's oral or written response had to satisfy in order to receive a particular grade. Thus, for example, in 1977, a letter from the Ministry of Education of the RSFSR recommended that an exam consisting of five problems should receive a grade of two "if the solutions to three (or more) problems contain crude mistakes (one or more)" (Chudovsky, Somova, and Zhokhov, 1986, p. 6). A grade of four, on the other hand, should be given if "the work has been fully completed and contains no crude mistakes, but contains small mistakes or more than two minor deficiencies, or small mistakes and minor deficiencies; [or] if four problems are solved without mistakes, but one problem either is not solved or contains mistakes" (p. 6). In addition, the letter explained which mistakes ought to be considered crude, which mistakes ought to be considered small, and which mistakes should be regarded as minor deficiencies.

Naturally, not everything in these criteria was precisely formulated; but even writing down some criteria proved sufficiently useful. Thus, for example, the guidelines underscored the fact that grades could not be lowered because students had made notations and then erased them or crossed them out in doing their work; such erased or crossed-out notations, it was pointed out, indicated that students were thinking and could not serve as grounds for lowering their grade. Indeed, not a few teachers would give students a grade of two for poor behavior in class and other such offenses or, on the contrary, would give a grade of five for arranging display cases in the mathematics classroom and so on, which distorted the overall picture of the students' success rate.

As will be discussed in detail below, during the school year a student receives grades on both tests and quizzes as well as for oral responses in class and so on. These grades are recorded in a special journal and determine each student's final grade for the class. In grades 5–9, final grades are given at the end of each quarter (the first quarter is September to October, the second quarter November to December, the third quarter January to March, and the fourth quarter April to May). In grades 10–11, final grades are given every half-year: the first half-year covers September to December, and the second half-year January to May. Some schools receive special permission from the Committees on Education to divide the school year into trimesters. The idea that the final grade should represent the arithmetic mean of all the grades received by a student has been criticized, but this arithmetic mean has in fact typically determined the final grade in practice. Very often, even very strong students have been prevented from getting a grade of "excellent" for the quarter if they received a single grade of two during the quarter. On the other hand, a large number of twos could also prevent a student from getting a grade of "satisfactory," even a fabricated one. Many teachers have resolved this issue by writing "pencil twos" in their journals and subsequently eliminating them with the help of an eraser. The term "concealed two" or "covered two" was widely used: a two would be concealed under a three (usually inflated), as students would be given repeated opportunities to retake tests or rewrite answers that they had failed to learn earlier.

We have already indicated that starting at a certain point there were practically no failing students in the USSR. But special note was also taken of students who had achieved high levels of excellence, particularly the so-called "medalists" — students who had graduated with a gold or silver medal. According to established rules, gold medals were given to those who had fives as their final grades for all of the half-years in grades 10 and 11, as well as on all of their exams. Silver medals were given to students with slightly less perfect records, allowing for one or two fours. Until recently, medalists enjoyed certain privileges in entering institutions of higher learning. In Russia (USSR), students were normally admitted to such institutions on the basis of entrance exams; medalists had to take only the first exam and would be admitted if they received a top grade (medalists who failed to get a top grade on their first exam had to take the other exams with the other students). For a school, having a large number of medalists has traditionally been a special point of pride. Consequently, schools have often taken deliberate measures to increase their numbers, making sure that a potential medalist is not given a two by accident and so on. According to our observations, during the 1990s the number of medalists increased by several times. Today, medalists do not enjoy any privileges in entering institutions of higher learning and are admitted to such institutions in the same manner as all other students, based on results of the Uniform State Exams (USE; see below). Unfortunately, however, the "spirit of competition" has continued to produce negative effects to this day.

Mention must be made of a crucial feature of the assessment process as it has developed in Russia (and the USSR). During all periods in the history of Russian schools, grades have been given *publicly*. A grade must be announced by the teacher in front of the whole class, both after students have been orally questioned at the blackboard and after their written work has been checked and corrected. In Soviet times, "grade screens" hung in classrooms, on which all grades received by students were displayed, with unsatisfactory or outstanding grades highlighted. At one time, the slogan "Learning is not your personal business" was quite widespread, and the education of each student was reviewed not only by teachers but also by the Komsomol organization of the class, and even by the Komsomol (Young Communist League) organization of the school. The practice of making students' grades public has continued to this day.

In connection with the fact that grades are made public, the "spirit of competition," which we have already mentioned, has been variously encouraged and promoted in the schools. Unfortunately, the frequent outcome of this has been not so much competition with respect to learning as competition with respect to grades. It should be kept in mind, too, that the fact that grades are made public — although undoubtedly traumatic for students in certain cases — is something absolutely ordinary and familiar to them; for this reason, they perceive this practice in a different way than it would be perceived, say, by American students if it were suddenly introduced in an American classroom. The fact that grades are public and that students can compare their results with those of other students ensures that grades remain objective and also facilitates the development of the students' ability to assess themselves. Not infrequently, teachers discuss the grades they give with a student or with the entire class, thus making their demands more precise and giving students some opportunity to contest those grades as well.

3 On the Nature of the Assignments Used for Assessment

For a long time, a fundamental feature of the Russian assessment system was that it eschewed multiple-choice tests. The attack on the discipline of "pedology" — the study of children's behavior and development — in the 1930s imbued the very word "testing" with negative connotations. The objectivity of multiple-choice tests, i.e. the independence of grades based on such tests from the judgments of the individuals administering them, could also not be held up as a virtue, since "bourgeois objectivism" was denounced in the methodologies of virtually every discipline as an approach that actually masked class interests. Today, many will agree that assessment based on multiple-choice tests cannot be seen as a completely objective approach or as an approach that does not unfairly privilege certain groups of students over others (Wilson, 2007).

Short-answer problems were also used relatively infrequently. Typically, solutions to problems require not only a detailed answer but also complete explication and substantiation. In certain respects, the forms of problems which were chosen for inclusion on tests were influenced by the conditions under which teachers had to work; for example, in the absence of copying technology, it was impossible to assign problems in a large number of different versions. Assigning short-answer problems when these problems were given only in one or two versions written on a blackboard would have been unwise, since cheating and copying other students' answers would have become too easy. Arguably, however, a much more important consideration was the conviction that a short answer provided no opportunity to assess the depth of a student's comprehension of a problem and could attest only to its superficial, "formal" understanding. Consequently, it was believed that only a detailed textual solution could attest to a student's genuine comprehension. In what was essentially an instructional article, one methodologist put it as follows: "The solution to certain problems should be accompanied by a detailed textual explanation, which should in essence constitute an essay on a mathematical topic" (Printsev, 1951, p. 72).

The degree of detail that such explanations could attain was illustrated, for example, by the way in which authors of another instructional article proposed formulating the final answer to the following problem: "Solve the inequality $4x^2 + 16x + 7 > 0$." Their version of the final answer ran as follows: "Given the expression $4x^2 + 16x + 7$, if we replace x with any number smaller than $-\frac{7}{2}$ (such as $-4, -5$, etc.) or any number greater than $-\frac{1}{2}$ (such as $1, 4$, etc.), we will obtain positive values" (Gurvits and Filichev, 1947, p. 46). Note that this was only the final answer to the problem — it was preceded by a detailed solution. It is not surprising that writing down the solutions to four or five exam problems could take hours (which is, in fact, how much time students were given to complete an exam).

In subsequent years, such excesses came under attack and a far more balanced approach was recommended (Dorofeev, 1982). However, to this day, the demands that must be met in the so-called "formatting of the solution," i.e. its presentation and exposition, have usually been quite stringent. As a result, they very often lead to arguments. Periodically, for example, one hears the extremist view that every line in the solution of an equation or an inequality must be accompanied by some kind of explanation — making it clear, for example, why this line is equivalent to the one that precedes it (i.e. why no roots are lost or gained in the process). A student's grade might be lowered because he or she failed to specify the domain of the expressions encountered in an equation, although these expressions were linear or quadratic and thus defined for all real numbers. The stated motive for lowering the grade in this case is that if the expressions had been more complicated — if they had contained radicals, for example — then failure to investigate their domains could have undermined the entire solution.

The ambition to develop students' mathematical communication skills and to assess the degree of their development, as well as the ambition to develop their ability to understand and substantiate a solution — not merely to memorize it as a routine procedure — are both highly commendable. But these ambitions can be successfully realized only in the presence of well-prepared teachers who are capable of exercising sound judgment in selecting problems and evaluating the completeness of their solutions. Since assessment relies on the judgment of the individual who corrects the tests, it is vital to have in place well-developed procedures for engaging in discussions with students and giving them the opportunity to contest their grades. The recently introduced Uniform State Exams (USE) are graded by three experts, in order to reduce the influence of subjective opinions. Appeals commissions have also been established. Theoretically, students can turn to them to contest not only a grade received on an exam, but also their current grades in school (as far as we can tell, however, this very rarely happens).

In the sections that follow, we will provide many examples of problems traditionally used in Russia for the purposes of assessment. On the whole, they naturally correspond to the problems found in textbooks; for this reason, both tests and quizzes contain numerous problems involving proofs. Such problems are typical of assessment in geometry, but this field is not by any means the only one in which they are encountered. For example, in a collection of pedagogical material published by Ziv (2002), a section intended for use in eighth grade in standard public schools contains the following problem: "Prove that, for any a, $a^2 + 3 > 2a$" (p. 8).

Let us repeat that the number of problems on a test, quiz, or exam has usually been (and largely remains) very small, but practically every one of these problems is multistepped. Attempts have also been made to construct tests and exams on the basis of blocks of interconnected problems as a way to permit students in some measure to use the solution of one problem to check the solution of another; to generalize the solution of one problem in solving another; or simply to use transformations and computations made in the course of solving one problem to solve the problem that comes next (Karp, 2003). For example, the following assignment was offered to a class with a so-called humanities specialization:

> Given the function $f(x) = 3^x$, a) solve the equation $f(x) = f(2x+1)$, b) solve the inequality $f(x) - f(2x+1) < 0$, and finally (c) construct a graph of the function $y = \frac{f(2x+1)}{f(x)}$ (Karp, 2003, p. 55).

Russian problems can sometimes be criticized for a certain artificiality and especially for their abstraction. Indeed, one might also point out that in Russian (Soviet) assessment, relatively little attention is devoted to what in the West are called "real-world problems." More precisely, if, until the end of ninth grade solving word problems is considered one of the most important testable skills, then after ninth grade the attention paid to them diminishes significantly (at least this has been the case during the last half-century). Without delving into the degree to which the word problems examined in school textbooks — both in Russia and, for example, in the United States — can be considered "real-world problems," let us note that it is far more difficult to pose a meaningful and, at the same time, not-very-difficult problem with real-world content based on material drawn from the upper grades in a Russian school. The following classic geometry problem given on a secondary school final exam in 1981 (Chudovsky *et al.*, 1986, p. 105) may be considered such an example:

> 24 dm^2 of material must be used to build a closed box with a square bottom. How long should the bottom's side be in order to build a box with the greatest volume? (The side must be no longer than 3 dm and no shorter than 1 dm.)

On the other hand, attempts were made in upper grades with a humanities specialization to return — in the teaching process, and therefore also in the assessment process — to word problems that could be reduced to linear or quadratic equations or to calculations involving percentages and the like. The following is an example of such a problem, which appeared on an exam (Karp, 2003, p. 94):

> The cost of participating in a conference consists of travel expenses (50%), food and housing expenses (31.25%), and a registration fee (18.75%).
>
> (a) Determine what percentage of the travel expenses is equal to the food and housing expenses.
> (b) Let travel expenses be 500 units greater than the registration fee. Determine the total cost of participating in the conference.
> (c) Use a bar chart to represent the relationships between the different types of expenditures for participating in the conference.
> (d) In order to cover conference expenses [see assignment (b)], a participant has received a grant in the amount of 1800 units. He spends the money left over on souvenirs. Use a pie chart to represent the distribution of his expenditures (compute the angles of the corresponding sectors).

We began this section by noting that multiple-choice tests were for a long time rejected by Russian education; but starting at a certain point (approximately since the 1970s), they gradually penetrated Russian schools, for example under the aegis of the once-fashionable programmed learning. They became far more popular, however, during the much more recent, post-*perestroika* period (thus, the already-mentioned USE, which will be discussed below, included both multiple-choice and short-answer questions along with traditional Russian problems). Like everything else that was once forbidden and then permitted, multiple-choice tests have become quite fashionable, although their quality has not always been high.

Among the first multiple-choice tests that were adequate for use on a large scale were P. I. Altynov's tests in algebra and geometry (1997). The problems on these tests were given only in two versions and at the same time were not difficult, so apprehensions about students copying one another's answers were not unfounded: good students who were not given problems that were difficult for them could quickly move on to distributing their answers. Below are two problems (presumably the easiest and the most difficult) from one of Altynov's tests, both of them dealing with elementary trigonometric equations and inequalities. The test contains 10 problems in all.

1. Solve the equation $\cos 0.5x = -1$.

   (a) $x = 3\pi + 4\pi n, n \in Z$; (c) $x = \pi + 2\pi n, n \in Z$;
   (b) $x = 2\pi + 4\pi n, n \in Z$; (d) $x = 0,5\pi + 0,5\pi n, n \in Z$.

10. Solve the inequality $\sin x > \cos x$. In your answer, indicate the sum of the natural numbers that are less than 10 and satisfy the given inequality.

    (a) 17; (b) 30; (c) 6; (d) 23.

The author established the following criteria for grading: a five for 9–10 correct answers; a four for 7–8; a three for 5–6; and a two for 4 and fewer. He recommended giving students one-and-a-half hours — two combined classes — to complete the test. However, the advisability of devoting two class periods to this work is open to doubt, because a good student will complete the given assignments in approximately 30 minutes while a weak student might never complete them at all, if only because problems that involve trigonometric inequalities go beyond the bounds of the standard public school curriculum.

Let us turn to examples of more difficult multiple-choice tests in geometry (plane geometry) for students in grades 7–9 in classes with an advanced course of study in mathematics (Zvavich and Potoskuev, 2006a, 2006b, 2006c). The book in which these tests appeared contained subject tests, survey tests, and summary tests. Each of them consisted of 16–17 problems in plane geometry and each was meant to be completed in 40–45 minutes. The authors recommended giving two fives for 16 correct answers, one five for 14–15 answers, a four for 12–13 answers, and a three for 9–11 answers. They also considered an alternative grading method, in which students would receive two points for every correct answer, have one point deducted from their total for every wrong answer, and be given no points if they left the answer to a question blank. With this approach, the scores would range from −16 to 32. Such a grading method is likely to eliminate meaningless guesswork, but in our view it is too complicated for both students and teachers. Below are three problems from a subject test devoted to the

Pythagorean theorem (note that problem 15 is marked with an asterisk, indicating a higher level of difficulty).

1. The points K, M, T, and P are located on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ of the square $ABCD$, respectively, in such a way that $AK = 3$, $KB = 5$, and $BM = CT = DP = 3$. Find the area of the quadrilateral $KMTP$.

   (1) 34; (2) 36; (3) 49; (4) 53; (5) 16.

14. Find the distance from the center of a circle with radius 4 to any chord of the circle of length 4.

    (1) $3\sqrt{2}$; (2) $2\sqrt{3}$; (3) 2; (4) 3; (5) 1.

15*. The diagonal of a right trapezoid divides this trapezoid into two right isosceles triangles. Find the length of the center line of this trapezoid if the length of its diagonal is equal to $2\sqrt{2}$ (Fig. 1).

    (1) 1; (2) 2; (3) 3; (4) 4; (5) 5.

As can be seen, these problems are sufficiently traditional for an in-depth Russian school course in plane geometry. Thanks to the multiple-choice format, many of the usual requirements are eliminated (explanation and substantiation, mathematically exact notation, the construction of an exact diagram, etc.). Students who have completed a large number of these demanding problems within the allotted time should hardly be suspected of having assimilated this material in a merely formal and superficial manner, and they should hardly be required to demonstrate their understanding by indicating all of the theorems on which they relied in solving these problems. At the same time, of course, it is much easier to correct and grade such tests than the more traditional variety.

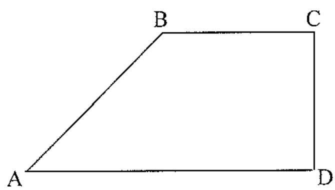

Fig. 1.

Ryzhik (2009) published a collection of problems aimed at combining the convenience that multiple-choice questions have for the purpose of grading and the openness and even ambiguity of traditional Russian problems. In each of these problems, students must write down one of the following signs: "+" (correct), "−" (incorrect), "0" (I don't know), "!" (the problem is not formulated correctly, since the object that is discussed is not defined), and "?" (no unequivocal answer can be given; additional information is needed). Students receive one point for every correct answer, lose one point for every incorrect answer, and get no points added or subtracted for writing down "0." Consider the following example:

> The graph of $y = f(x)$ passes through the origin. The graph(s) of which other function(s) also pass(es) through the origin?
>
> (1) $y = 2f(x)$; (2) $y = f(-x)$; (3) $y = f(x+1)$;
> (4) $y = |f(x)|$; (5) $y = 1 - f(x)$. (p. 136)

In this case, it is clear that the answers to questions 1, 2, and 4 must be "correct" ("+"), the answer to question 5 must be "incorrect" ("−"), and the answer to question 3 must be "?" since it is not known what value the given function has when $x = -1$.

An example of a situation in which the right answer is "!" may be seen in a problem that asks students to indicate whether the number a is irrational, given that $a = (\sqrt{-2})^2$ (p. 21). Since complex numbers have not yet been studied at this stage, such a number does not exist.

Less subtle problems, however, have received the greatest attention. Problems from the USE have had a serious impact, and of 26 problems typically offered, 10 were multiple-choice problems. Two examples of such problems from the 2007 version of the USE are (http://www1.ege.edu.ru/content/view/21/43/):

1. Find the value of the expression $4^{6p} \cdot 4^{-4p}$ when $p = \frac{1}{4}$.
   (1) 1; (2) 2; (3) 32; (4) 4.
2. Simplify the following expression: $\frac{\sqrt[3]{54}\cdot\sqrt{16}}{\sqrt[3]{250}}$.
   (1) 1.2; (2) $\frac{6\cdot\sqrt[3]{2}}{5}$; (3) 2.4; (4) $\sqrt[3]{2}$.

It should be noted that multiple-choice problems on the USE have provoked strong criticism (for example, see the website http://www.

mccme.ru). In particular, it has been argued that although computational problems similar to the ones that appeared on the exam have a role to play in teaching, since without acquiring certain skills students would be unable to solve any substantive problems, they are misleading on a final exam and appear useless, pointlessly artificial, and old-fashioned. One may expect that the problems on the USE will undergo certain changes in the future.

In conclusion, it should be noted that college entrance exams have also had a considerable influence on secondary schools. Such exams have employed a great variety of problems: although the dominant format has remained the traditional Russian schema of five or six problems to which students are required to provide detailed and substantiated solutions, other kinds of problems have long ago begun appearing on the exams as well, including multiple-choice questions.

4 Oral Questioning in Class

In Russia, traditionally, grades are given not only for students' written work but also for their oral work. Many forms of class participation are graded, and below we will name only the most important among them. Russian pedagogy places much importance on oral questioning (it is a telling detail that, in the past, inspectors checking a teacher's work usually remained displeased when they saw that the teacher's journal contained only columns of grades for tests and other written work: the journal was expected also to contain sporadic grades, i.e. grades for only some of the students in the class — for oral work). In addition to serving as a natural way of testing a student's knowledge, oral questioning constitutes a powerful means of developing students' speech in both broadly educational and narrowly subject-oriented terms. Students' oral responses can be structured both as a monolog and as a dialog with the teacher or with other students in the class. It is assumed, of course, that students listen carefully to their classmates' answers and actively participate in the discussion of what has been said or done by them. Teachers take various measures to sustain the class's attention in this way, first and foremost by posing questions to the whole class after a response is given by one of the students.

Below, we discuss separately the two alternative cases of a response given by a student "from his or her seat," and the usually more elaborate response given "at the blackboard." It should be emphasized, however, that what we consider to be the strongest feature of the methodological style that has developed in Russia — present in both types of oral questioning examined below — is the teachers' ability to pose various kinds of additional questions, questions that deepen, expand, and explain what has been done. A student who has explained the solution to a problem may be asked questions about the justification for one or another step in the solution, about how the solution would have to be changed if the formulation of the problem were also changed, and even about how the problem might be formulated for some other object. By examining the problem from different angles, the teacher and the student create a kind of model for how a problem-solver should approach it, on the one hand, while on the other hand the teacher is given an opportunity to assess the depth of a student's awareness and grasp of the material.

4.1 *The "From the Seat" Response*

The most widespread form of oral work in class is the so-called "from the seat" response, in which a student briefly answers a question posed to the class or offers a response on his or her own, after asking permission to speak, as part of a class discussion. The response may be very brief or even monosyllabic. For example, in conducting oral computations, the teacher might first assign some problem and then ask the class for their results. In such cases, the teacher usually asks several students in a row for their answers, and only after determining that all or at least many of them have obtained the correct result does the teacher proceed to a more in-depth discussion.

A "from the seat" response, of course, may also be requested in more complicated situations and may set the stage for other forms of oral or written work. For example, while discussing the topic of "relative location of planes" in a 10th-grade classroom, a teacher may ask the class to describe all possible ways in which three planes in space may be positioned with respect to one another. After giving the students a minute or two to think about it, the teacher might call on one of them and ask this student to list all of the possible cases orally, and then call on a second student to sketch all of these cases on the blackboard.

The "from the seat" response is an essential part of general discussions, which a teacher may organize, for example, when embarking on some new topic and the students are just developing a basic understanding of it. Thus, sketching several different quadrilaterals on the blackboard, a teacher might single out some of them, indicating that these particular kinds of quadrilaterals, known as parallelograms, will be studied in the next few lessons; after which the teacher might ask the students themselves to come up with a definition for a parallelogram.

To be sure, in this kind of situation, it would hardly be useful to give a formal grade for a wrong answer (although it would be quite natural to reward a good answer with a formal grade). Indeed, formal grading is by no means necessary in any of the other described examples either. At the same time, teachers often keep track of how actively a student participates in class by making special marks in their journals and then giving a grade based on several responses. In any event, by working with the class in this way, a teacher acquires a better understanding of the students, and the students themselves come to see their own difficulties and strengths more clearly.

"From the seat" responses come in many different forms, and sometimes they can only nominally be classified as part of oral work. To give one example: all students in a class are given one or another "tricky" question [such as: construct the graph of $y = (\sin x)^{\log_{\sin x} 2}$], and the teacher asks the first two students who have completed the assignment to come up to him or her and show their answers. If a student's answer is incorrect, the student does not receive a bad grade, but loses the possibility of coming up and showing the teacher the answer to this question. The first student who shows the teacher a correct answer receives a five.

Mention must also be made of specific psychological problems connected with "from the seat" responses and with students' levels of class participation in general. In moving up from one grade to the next, students' attitudes toward voluntary "from the seat" responses change. If in grades 4–6 (not to mention of grades 1–3) students constantly raise their hands and want to answer every question, sometimes regardless of whether or not they know the answer — or whether or not they even heard the question, then starting in seventh grade this "forest of hands" begins to thin out catastrophically until, by 11th grade, often only two or three students at most raise their hands, while many students who know the correct answer to a question do not. Some do not raise their hands because they are not sure of the correctness of their answer and are afraid that the teacher will ridicule their answer, or that their classmates will laugh at its incorrectness. Others do not raise their hands because they are afraid that if they answer correctly and the teacher praises them, their classmates will smile ironically and view them as "social climbers" or "too clever." For this reason, teachers are often forced to call directly on respondents, without looking at which students have raised their hands; this is also useful, however, because it allows teachers to involve those students in class discussions who might have preferred to sit quietly without participating if left to their own devices.

4.2 *The "At the Board" Response*

Until almost the end of the 20th century, the response at the blackboard or next to the teacher's desk was a crucial feature of classes in Russian schools. During the 1970s and 1980s, it was common to see 5–10 students being called up to the board simultaneously, for which purpose many mathematics classrooms had blackboards not just on the front wall but also on the side and rear walls. Almost a third of the students in a class might be out of their seats, and those who remained seated would constantly rotate their heads to see the student giving the response. It appears that, gradually, such large-scale, head-on approaches to questioning students became less popular. As early as the 1960s, many began to favor the so-called Lipetsk method (named after the experience of teachers in the Lipetsk region), which effectively consisted of activating every student in every class and giving grades to every (or practically every) student for their work in class (Moskalenko, 1959). This method was energetically promoted and virtually forced on all schools, which naturally led to its demise.

Currently, teachers by no means always use the "at the board" response during every class. However, in mathematics classes, this technique is still employed frequently, most often for publicly solving problems on one topic or another. Below, we list several characteristic situations in which "at the board" responses are used.

4.2.1 *Going over homework assignments*

To go over homework assignments, teachers usually call up several students to the blackboard, each of whom must use some portion of the board to solve one exercise or another from the homework assignment and to explain how it was done. There are two schools of thought among teachers regarding the form in which such "at the board" responses should be given. Some believe that students must come up to the blackboard without their notebook that contains the homework assignment (or leaving their notebook on the teacher's desk) and solve the problem on their own. Such an approach, in the view of its supporters, ascertains whether the student completed this homework assignment on his or her own and to some extent discourages students from copying their friends' answers or using "solution books" to prepare their homework (i.e. books that contain solutions to problems from common textbooks and problem books — such "solution books" are now published in considerable quantities and vary in quality). Other teachers are convinced that the answer will be much more precise and compact if, on the contrary, students are permitted to use their own notebooks, copying their solutions from these notebooks directly onto the blackboard. In this case, the process of copying takes up little time, and the bulk of the student's response consists in the clear and precise oral substantiation of the solution. Teachers are persuaded that even if the student has copied the homework from a friend or relied on someone's assistance, but has nonetheless grasped the solution and is able to explain and substantiate it competently, then this, too, deserves to be valued sufficiently highly.

4.2.2 *Questioning students about theoretical material*

Traditionally, a lesson in Russia began with a student being called up to the blackboard to prove a theorem that had been proven during the previous lesson. Naturally, such responses often relied on rote learning, and it is no accident that the memoirs of 19th century writers so often mention the cleverness of a teacher who had, for instance, rearranged the letters that indicated the vertices of a triangle, thus catching everyone off-guard (Karp, 2007b). Yet, the opportunity to hear and talk about a given proof one more time, as well as the chance to carry it out in front of others, was very beneficial to students' mathematical development. Today, the inadequate time allotted to the study of mathematics in the Russian school curriculum makes it impossible for teachers to systematically devote to theoretical material the attention that it deserves. Nevertheless, students' knowledge of theoretical material is still tested orally in class. Students are called up to the blackboard and asked either to prove one theorem or another, or to analyze one theoretical issue or another from beginning to end. Most often, students are given such tasks in geometry classes (for example, while studying parallelograms, one student might be called up to the blackboard to prove that the opposite sides and angles of a parallelogram are congruent, while another might be called up to prove that a parallelogram's diagonals bisect each other). However, even in algebra or calculus classes, one comes across students being questioned about theoretical matters. For example, a student might get called up to the blackboard to prove the so-called Viète's theorem about the relation between the roots and coefficients of a quadratic equation or to talk about the properties and the graph of the function $y = x^3$.

The student usually spends about five minutes preparing the answer, and the answer itself lasts from five to ten minutes, depending on the question and the level of the student's knowledge. A questioning session of this kind may come as a surprise for the students in the class (i.e. the students may have been asked to learn the proofs, but not told when they would be required to talk about them), or it may come after being announced during the previous lesson. Moreover, sometimes during the previous lesson, the teacher will have listed the names of several students who might be called up to the blackboard during the next lesson to answer questions about tasks already known to them. Students' responses to questions may also take the form of "reports" on assigned topics which are prepared in advance.

While one student is responding at the blackboard, another student might be given a chance to prepare for an answer at his or her seat. In certain cases (particularly in the case of a relatively lengthy report), a student may be permitted to use a plan or even a summary of the answer, prepared in advance and written down on a sheet of paper or in a notebook. Based on our observations, the report approach is by no means always successful and sometimes results in the student monotonously reading a prepared text, often something downloaded from the Internet.

While a student is giving a lengthy oral response to a question, the other students in the class must listen attentively. To engage their attention, the teacher may ask them questions while the answer is being presented, or they may be asked to provide oral or written responses to the presentation they have heard. At the end of the presentation, the students in the class, as well as the respondent himself or herself, may be asked to grade the response and then compare it with a grade proposed by the teacher. After a grade is given, it is useful to ask respondents whether the grade makes sense to them, and if not, to explain it to them.

4.2.3 *Solving problems on the blackboard*

An enormous portion of the time devoted to studying mathematics in Russian schools is spent on problem solving. As a rule, teachers explain the theoretical material in class or tell students to read about it in a textbook; only relatively rarely do they question students about it. Problems are a different matter. The art of teaching manifests itself, in this instance, not in the way in which teachers themselves solve problems on the blackboard, but in the way in which they organize problem solving by the students. Solving a problem in class differs in significant ways from solving a problem independently with a problem book, if only because the teacher can unobtrusively help the student and direct him or her.

Solving problems generally means moving along an unknown path, and therefore the student can naturally stop in order to think things over. On the other hand, if the student thinks for too long, the lesson will be lost. There are various approaches to dealing with this difficulty. One method is as follows. Student A is called up to the blackboard to solve a sufficiently easy problem (in the teacher's opinion, this student should be able to handle this problem). At the same time, students B and C are given problems, told to read them, and asked to prepare answers at the blackboard. While student A spends three or four minutes solving his or her problem together with the class, student B has time to get a sense of the solution of his or her more difficult problem and go up to the blackboard, if not with a solution in hand, then with its plan; at the same time, a fourth student, D, will receive his or her own problem and get prepared while student B, and then student C, give their solutions. Ideally, each problem should be selected to match the level of the student to whom it is assigned: students should be capable of solving the problem they are given. Other students in the class may be drawn into solving a problem by being asked to suggest their own approaches and to correct the mistakes they have noticed. Such a process is virtually impossible to program in advance; to organize it, a teacher must not only grasp the conditions and solution of the problem, but also quickly determine which of the student-suggested approaches to solving it are incorrect and which are correct, and which of the correct approaches are rational and which are irrational; the teacher must give the students the freedom to be creative and at the same time lose neither the time nor the thread of the lesson.

Giving grades to students who are solving a problem on the blackboard is also a delicate matter. Quite often, teachers refuse altogether to give formal grades for solutions to new problems presented on the blackboard, believing that the threat of getting a bad grade becomes a source of stress for students and makes it difficult for them to think. An informal grade, however, is present in any case, since the problem's solution is either accepted or not accepted by the teacher and the class. There exists an opposing point of view, according to which a grade that is clearly stated by the teacher is useful both for the class and for the student.

Traditional Russian methodology required that students not simply solve a problem on the blackboard, but also comment on and explain their actions (for example, that they point out the equivalence of various equations or identify those properties of functions which are used in the solution, etc.). Today, in many classes, one can see teachers themselves providing necessary commentary or asking students for explanations once the solution has been written down, recognizing that students should not be interrupted while solving an unknown problem.

5 Written Work

The forms and methods used for written assessment are many and varied. Below, we list the basic types.

5.1 *Tests*

In Russian schools, tests in mathematics are strictly mandatory. Their number is regulated and teachers may be censured for either decreasing or increasing this number. As a rule, three tests in algebra and two in geometry are given in one quarter. Traditionally, teachers have been advised to use special test notebooks for tests (rather than separate sheets of papers), and to preserve these notebooks at least until the end of the school year, so that the vice-principal, the students' parents, or a mathematics supervisor visiting the school might examine how well the tests were written, how competently they were checked and graded, and what mistakes were made by one or student another. Test grades are highlighted by many teachers in their journals in a special way and play a decisive role in determining the students' overall grades for each quarter.

In writing tests, teachers choose their own strategies. Some teachers use their own problems on tests (problems either invented by them or drawn from various textbooks and problem books). They are usually either beginning and inexperienced teachers (whose tests are often not very good) or, on the contrary, experienced teachers with many years of work behind them (and their tests are, as a rule, good). Most teachers use all kinds of collections of tests and educational materials. Since the

1960s, the number of such collections has grown rapidly and both mathematics supervisors and teachers have written them. Note, too, that for a long time almost all educational materials were submitted for review to the Ministry of Education of the USSR, and then of Russia, and were labeled as "Recommended by the Ministry." In the 1990s, this procedure became optional, and subsequently it came to be considered as altogether unnecessary. Currently, textbooks alone are submitted for review to the Ministry.

Naturally, using tests from published collections unthinkingly is by no means always effective, since the tests offered in these volumes are created for abstract students, while a teacher must deal with a concrete class and its idiosyncrasies. Many teachers alter somewhat the material on the published tests, making them more difficult or simpler. In the latter case, there is a danger of making them so simple that students' grades on these tests will not correspond to any generally recognized level (many attempts have been made to define this level formally — see for example Firsov, 1989 — and efforts to create exact standards in mathematics, i.e. to provide a precise description of the requirements, have continued to this day).

The authors of collections of tests have varying views about how such collections ought to be compiled. Two points of view, which are in some sense polar opposites, may be singled out. One view holds that a published test should be highly compact and oriented toward the overwhelming majority of the students in a class. Students' grades on the test will naturally vary, but the authors do not consider it necessary to assign problems at different levels of difficulty. According to the other view, a test must contain several levels (at least two). Consequently, students may be given multiple problems on the same topic, but at different levels of difficulty. Predictably, such an approach frequently means that a test will have superfluous problems, in the sense that students might get a five on the test even if they have not completed all of the problems. Such an approach makes teachers more aware that the goal is not merely to satisfy the minimum requirements, but also to strive to attain the maximum that each student is capable of. At the same time, it may be noted that with such an approach weak students will encounter problems on tests that are fundamentally incomprehensible to them; thus, although solving these problems is by no means necessary for passing the test or even receiving a good grade on it, generally speaking this might produce a certain psychological discomfort.

Let us give an example of a test whose authors may be characterized as moderate adherents of the second point of view. This test is intended for seventh graders, to be completed in one class period, and given after students have studied equations and word problems that are reducible to linear equations. Those problems on the test which belong to the so-called "required level" are marked with the symbol •. The authors leave the criteria for grading up to the teacher's discretion — pointing out, however, that the teacher may decide to give a five both when all problems have been solved and when one of the last two has not been solved.

1. • Solve the equation (a) $3x+2.7=0$; (b) $2x+7=3x-2(3x-1)$; (c) $\frac{2x}{5}=\frac{x-3}{2}$.
2. • Three seventh grade classes contain a total of 103 students. Class 7b has four more students than class 7a and two fewer students than class 7c. How many students are in each class?
3. Solve the equation $\frac{2x-1}{2}=\frac{x+5}{8}-\frac{1-x}{2}$.
4. In three days, a hiker walked 90 km. On the second day, he walked 10 km less than on the first day, while on the third day, he walked $\frac{4}{5}$ of the distance that he covered on the first and second days together. How many kilometers did the hiker walk every day? (Zvavich, Kuznetsova, and Suvorova, 1991, p. 104)

As we have already remarked, checking and grading a student's work by no means consists in merely counting the correct answers. The teacher usually first evaluates each problem individually, often making use of the following system of classification and symbols:

$+$ The problem has been solved correctly; no comments.
$\pm$ The gist of the solution is correct, but there is a small mistake.
$\mp$ The problem has been solved incorrectly, but certain knowledge and skills have been demonstrated.
$-$ The problem has been solved incorrectly.
0 The student did not attempt to solve the problem.

> Some teachers add a set of more subtle categories:
> +. The problem has been solved correctly, but with an insignificant deficiency.
> +! A difficult problem has been solved exceptionally well and requires a special additional five (this test contains no such problems).
> −! The problem has been solved exceptionally poorly; the student has demonstrated a complete lack of comprehension of this problem.

The grade ± might be given, for example, if a student has correctly solved a given equation but has made a computational error (in a problem in arithmetic, of course, a mistake in arithmetic would cause a grade to be lowered more substantially). The grade ∓ might be given, for example, if in the solution to problem No. 4, the equation was formulated in a fundamentally incorrect way, but then solved correctly.

Subsequently, the teacher usually analyzes the results obtained and establishes criteria for grading (and different teachers' opinions about the lower and upper bounds of each grade do not necessarily coincide). We would consider it reasonable to give a grade of three for the test reproduced above if the results, for example, were as follows:

1a	1b	1c	2	3	4
+	+	∓	+	−	0

Another important factor that comes into play in grading tests, as has already been noted, is the level of detail that the teacher demands in a solution. Clearly, the more problems a test contains and the more difficult these problems are, the lower will be the level of detail that students can offer. If students are required, in solving a problem in geometry, to indicate at every step what theorems they are using in the course of their reasoning, then they will have time to solve no more than two, or at most three, problems. While it is certainly necessary to give such tests, it is not necessary that the problems on such tests be difficult. One of the basic methodological aims of such tests is to see how well students can express their ideas in written form. Other kinds of tests, however, are possible and necessary as well. For example, students may be given a test containing 8–10 or even more problems, sometimes with accompanying diagrams. Here, the students must "grasp the situation" and, after jotting down some straightforward computations or conclusions, write down the answer (such tests may make use of the multiple-choice format as well). We believe, therefore, that it makes sense to specify beforehand for each test what level of detail the solutions to the problems on it are expected to possess.

Regarding the issue of preparing for tests, one should say that no test must ever come as a surprise to the students. It is useful, for example, to hang up a schoolwide test chart, which helps, at least to some extent, to regulate students' workloads. According to established official rules, students may not be given more than one test per day. However, formally speaking, students may be given any number of quizzes per day, and indeed students may be questioned in all of the six or seven classes that they have on a given day — which undoubtedly can become an excessive burden for them.

Although "teaching for the test" should never be a teacher's goal, this does not mean that there should be no test preparation whatsoever: on the contrary, students should be taught to prepare for tests. Russian teachers employ various strategies for this; for example, the teacher can have the class identify the basic ideas of a topic that has already been studied, can remind the class of the most characteristic problems from the textbook and assign them as homework, or can compose a kind of preparatory test and assign it as homework or discuss it in class.

When a student gets a two on a test, many teachers in Russian schools give this student the opportunity to take the test over. In certain cases, this has been useful and has led to improved assimilation of a given topic by the student; in other cases, it has led to the student repeatedly retaking the test until a grade of three could be squeezed out of it. In some instances, the teacher would demand that students work on their mistakes either directly in their test notebooks or in their workbooks. Then, the teacher would check the students' work on their mistakes, and this work would be used to "cover up" the previously given grade of two.

Test scores are usually announced in front of all students. As part of this process, a student's mistakes are publicly examined and correct solutions are demonstrated. This gives students an opportunity to learn not only from their own mistakes but also from the mistakes of their classmates. Naturally, a great deal in such situations depends on the pedagogical skills and tact of the teacher, since without such skills and tact this approach — which is, generally speaking, quite useful — can lead only to psychological trauma. The famous pre-Revolution journalist Doroshevich gave a description of how students felt while waiting for a teacher to go over a written assignment that they had completed: “The day of the notebooks’ ‘return’ arrived. This was a day that we always waited for with particular impatience. We were in store for a whole hour of ridicule directed against our weakest comrades” (Doroshevich, 1962; p. 204).

In particular, it is important that a student understand that a teacher’s corrections on a test do not represent the last word and the final truth. If, for example, a schoolboy or a schoolgirl has noticed a mistake that the teacher has overlooked or a notation that has been underlined as a mistake but in reality is not and has turned out to be correct, then it is reasonable not only to correct the teacher’s mistake but also to give special encouragement to those who noticed it. Moreover, students should be commended for devoting attention to a corrected test even if their observations are not correct. A test is not only an occasion for testing, but also an occasion for teaching.

5.2 *Quizzes*

Here, we translate the Russian expression “*samostoyatel’naya rabota*” (literally, “independent work”) as “quiz,” but this is not quite accurate. In Russian, this expression simply means that students work without help from the teacher; in and of itself, it does not imply that this work must be graded. In actual practice in Russia, grades usually *are* given or may be given for such work, but it is still important to recognize this distinction, which suggests a broader use for the problem sets examined below.

Many Russian teachers conduct a 5–10-minute quiz during almost every lesson, with the aim of assessing the students’ assimilation of new material or material already covered. As a rule, such quizzes differ from tests not only in terms of the time allotted for them, but also in the manner in which they are conducted and graded. Students taking quizzes may use special notebooks, separate sheets of paper, or simply their own notebooks. When students take tests, they are strictly forbidden to use any reference books whatsoever and prohibited from relying on their own preparatory notes or crib sheets; when students take quizzes, on the other hand, they are often allowed to use any reference material they wish, with the exception, naturally, of copying from classmates. After a test, all of the students' notebooks are removed from them and checked; after a quiz, a teacher might select only a few notebooks for checking — although, of course, the decision can be made to check all of the quizzes as well.

It is telling that collections of material for assessment typically contain many more quizzes than tests. For example, the already-mentioned collection by Zvavich *et al.* (1991) contains 10 tests and 56 quizzes. A quiz may make use of material from a textbook or a regularly used problem book. In such cases, it is assumed that every student will have access to such a textbook or problem book during the quiz, and that these textbooks or problem books do not contain answers to the problems given on the quiz. The teacher presents two or more sets of problems from the book, of similar levels of difficulty, on the blackboard and asks the students to solve them either on separate sheets of paper or in their notebooks over a given period of time. After the students complete these quizzes, the work of some or all students is checked and graded.

Teachers' attitudes to quizzes vary. On the one hand, quizzes give teachers an opportunity to accumulate grades and systematically evaluate their students' knowledge. On the other hand, a quiz can consume at least one third of a class period, can lower the level of the teacher's oral interaction with the students, and additionally can require the teacher to spend a great deal of time on grading. Some teachers never conduct quizzes for the whole class, but there are others who conduct two quizzes per class. Many teachers use published quizzes in a different capacity, solving them in class in oral questioning sessions with the students, giving them to the students as homework assignments or calling up several students to the front desks (or some other specially designated place) and giving them these quizzes as "individual assignments."

As a rule, the lengths of quizzes vary from one problem to 3–5 problems. In contrast to tests, quizzes are often aimed mainly at teaching students rather than assessing them formally. Many authors publish quizzes in several versions with different levels of difficulty. The first and second versions of a quiz are usually easier than the third and fourth. Some collections contain even more versions and even more sophisticated differentiations. B. G. Ziv's problem book in geometry (1995) (which the author, however, titled *Problems for Classes* — thus avoiding the need to discuss how much time should be allotted to each assignment) offers eight versions of quizzes on every topic: versions 1–2 are intended for weak students, versions 3–4 represent the basic level, versions 5–6 are only for the most capable students, and versions 7–8 can be used in math clubs or "math circles" (p. 3). Problems from the first and third versions for eighth graders are given below; the topic is "Area of a Triangle" (p. 148):

Version 1

1. In the quadrilateral $ABCD$, $BD = 12$ cm. Vertex B is 4 cm away from the straight line $\overleftrightarrow{AC}$. Find the area of the triangle ABC.
2. In the triangle ABC, $m\angle C = 135°$, $AC = 6$ dm, and $\overline{BD}$ is the height, whose length is 2 dm. Find the area of the triangle ABD.

Version 3

1. In the triangle ABC, $m\angle B = 130°$, $AB = a$, $BC = b$, while in the parallelogram $MPKH$, $MP = a$, $MH = b$, $m\angle M = 50°$. Find the relation of the area of the triangle to the area of the parallelogram.
2. In the right triangle ABC, O is the midpoint of the median $\overline{CH}$ (H lies the hypotenuse $\overline{AB}$), $AC = 6$ cm, $BC = 8$ cm. Find the area of the triangle OBC.

As can be seen, such problems may be used both for assessment and for teaching. The ideas used in solving the different problems in some measure echo one another (for example, in both the second problem of the first version and the first problem of the third version, it is useful to find the angle that is supplementary to the one given). For this reason, systematically and independently solving the problems from the different versions helps to develop students' geometric vision and thinking.

5.3 *Mathematical Dictations*

During the 1960s–1980s, teachers in Russian schools often questioned their students in a written format called "mathematical dictation." In recent years, this approach has been significantly less popular, although it continues to be used. A mathematical dictation is usually devoted to a specific topic, is given in one or two versions, and requires students to write down on a special sheet of paper either a key word or a numerical solution, usually obtained orally after hearing an orally posed question. The questions are either read by the teacher or played on a tape recorder.

As an example, consider the dictation on the topic "Exponents" for sixth graders from the book *Mathematical Dictations*:

1. Write down an exponential expression with base 3 and exponent 2.
2. Write down in exponential form the product of four multiples, each of which is equal to b.
3. Write down the expression: ten to the fifth power.
4. Find the value of the fourth power of the number −2.
5. Find the value of the fifth power of the number −1.
6. Find the value of the sixth power of the number 1.
7. Find the value of the seventh power of the number 0.
8. Write down the exponential expression with base x and exponent 3 in the form of a product. (Arutiunyan *et al.*, 1991, pp. 21–22)

Sometimes teachers would not limit themselves to dictating the "dictations," but would also make use of overhead projectors or other technology to demonstrate some piece of information on the screen or the blackboard. This was done, for example, when the problems given to the students were associated with graphs or with geometrical material.

The fashion for dictations can be explained, on the one hand, by the teachers' desire to give their lessons and tests a more varied form, and on the other hand, by the encouragement that this fashion received from officials' demands that technological teaching aids be used in the classroom. Classes were crammed with overhead projectors, tape recorders, all kinds of revolving models, and so on. The use of technological teaching aids became an end in itself and, consequently, teachers no longer selected a teaching aid to match some important mathematical activity but, on the contrary, selected the mathematical activity in such a way as to use a teaching aid. Gradually, as has already been noted, the fashion for dictations waned.

5.4 *Individual Written Questioning of the Student in Class*

Traditionally, in Russian (Soviet) schools, lessons began with several students being called up to the front desks and asked to complete individual assignments. Usually, the students were given a clean sheet of paper to write down their answers and another piece of paper with an assignment. The assignment might be a version of a test on a previously studied topic, a version of a quiz on a previously studied topic or a topic currently being studied, a selection of problems or exercises (or even a single problem) on a given topic, and so on. Students were given a specific length of time to complete their individual work.

Teachers could make up the material for such assignments themselves or take it from published collections for use in schools or for preparation for college entrance exams [the most widely used among such collections is probably the problem book by Skanavi, (2006)]. The assignments could include problems of different levels of difficulty, so that, for example, each student would initially receive a piece of paper with one problem of the first level. If the answer obtained by the student was wrong, the student's individual sheet of paper would be marked with a minus sign and the student would be given a different problem, again of the first level. If the answer was correct, however, then this student would receive a plus sign and get a problem of the second level, and so on. This form of individual questioning could occupy an entire lesson. It is quite productive, but also quite demanding on the teacher, who must both prepare for such a session and exert himself or herself during it.

More often, however, teachers have limited themselves to giving their students a single assignment card, to be completed in 10–15 minutes, i.e. in the time allotted to checking homework assignments, regular questioning of the students, and so on. Such individual work with assignment cards enables teachers not only to assess the work of a larger number of their students, but also to some extent to achieve differentiation in the classroom. Stronger students receive more difficult problems while the rest of the class work on ordinary material (and, conversely, students who are having difficulty with ordinary assignments might be given material that is not too difficult for them).

The content of a written individual assignment does not necessarily consist of a set of problems; it might also include theoretical assignments (prove a theorem, provide a list of known propositions on a given topic, provide a definition, etc.). Other forms of assignments are described in the literature as well. At one time, assignments that required students to write out so-called "supporting conspectuses" were very popular. These conspectuses were invented by V. F. Shatalov (1979, 1980), a teacher of physics and mathematics, and they contained brief descriptions — in part graphic and symbolic — of the topics being covered in class. Every student would have to fill out — and receive grades for filling out — such conspectuses almost daily, which, according to Shatalov, ensured that the students would retain what they had studied and led to the disappearance of twos, and even of threes. At a certain point, Shatalov even presented his lessons on national television, which turned out to have positive consequences since it put an end to the view of Shatalov as a persecuted solitary genius and allowed educators to examine his methods in a more sober light (Dadayan *et al.*, 1988).

We should note that the term "questioning" also signifies simply eliciting students' opinions, for which no grades are given. The St. Petersburg teacher A. R. Maizelis (2007), for example, regularly handed out small pieces of paper during his classes and asked students to express their opinions about a whole range of questions: his students would be asked to generalize some observation made during the lesson and formulate some theorem, report a mistake they might have noticed, formulate some hypothesis, or even simply pose a question. Naturally, in some cases, students would get a five for making an apposite assertion (or even for posing a good question), but usually the aim of this exercise was not to produce a formal assessment but to offer several students simultaneously the opportunity to express their views in class.

6 Long-Term Assignments

Projects, which are popular in the United States and many other countries, were also popular in Russia (USSR) in the years following the Revolution. The changes in pedagogy that occurred in the 1930s forced educators to make a complete break with these techniques. The very word "project" did not re-enter the school lexicon until decades later. However, assignments meant to be completed over an extended period of time (a week, a month, a summer vacation, etc.) have been and continue to be used in Russian mathematics education and assessment.

Probably the most widespread assignments of this type are problem sets that students are given to solve. Such problem sets can be put together in the most varied ways. The teacher may simply ask the students to solve all of the problems on a given topic from some problem book. For example, school textbooks often contain sets of review problems at the end of each chapter; alternatively, teachers may compile a problem set themselves by drawing on problem books ordinarily used for quizzes and tests. Some collections or books for teachers contain special sets of assignments meant to be completed over an extended period of time — in particular, sets of difficult problems on various topics for classes with an advanced course in mathematics (Karp, 1991).

When teachers give students such assignments, they usually realize that it is practically impossible to guarantee that the students will solve them completely independently. Therefore, such assignments are often seen as having, first and foremost, an educational function rather than a formally evaluative one. Consequently, students are assessed on the basis of the solutions which they have provided and on the basis of their ability to reproduce some of these solutions in class meaningfully (in the context of an oral or written review).

A long-term assignment may also require students to study new material on their own — for example, to read a section of a textbook that is marked with an asterisk as optional and to solve the problems in this section. The assessment itself may be carried out, for example, orally, after class; in such cases, usually only good grades are given.

Long-term assignments may also be of a completely different nature. The already-cited A. R. Maizelis (2007) often had his students build various kinds of models. Building even an oblique triangular prism is not easy, and Maizelis's students built models that were far more difficult than that. These models were put to use in geometry classes and other, even nonmathematical, classes; at the same time, they served as a means of assessing the students' ability to think geometrically. Today, in addition to Wenninger's classic book (1974), one can recommend a number of newer texts to teachers who are interested in giving such assignments — for example, Zvavich and Chinkina, 2005. However, as far as we have been able to observe, assignments of this type are not widespread.

We have already mentioned student-prepared reports, such as those about the lives and work of research mathematicians. The preparation of such a report is, of course, also a long-term project, as is the writing of research papers in general. According to our observations, projects connected with the study of various applications of mathematics (for example, the collection of various kinds of data and the subsequent identification of various kinds of patterns in the collected data), which are popular outside Russia, are today quite rarely employed in Russian mathematics education, possibly because not much attention is devoted to topics in finite mathematics in general. As for projects that involve any kind of measurements, they are usually carried out within the framework of other subjects — above all, physics.

In recent years, a new form of assessment has begun to penetrate school education, namely the creation of a portfolio. The word "portfolio" does not exist in Russian (the Russian word is "*portfel'*") and its very use already reveals a deliberate borrowing from foreign practices. Lukicheva and Mushtavinskaya (2005), for example, propose the following structure for a portfolio in mathematics:

1. Official documents (for example, certificates from Olympiads and competitions);
2. Creative work (here, the authors suggest including records of the student's participation in activities that have no official status, reports and research papers, projects and models, as well as the student's best mathematics notebooks, written tests, and quizzes);
3. References, recommendations, and self-reports.

A student's portfolio, in the opinion of Lukicheva and Mushtavinskaya, must be repeatedly assessed by the teacher and by the other students, both through discussions about it and through a formal presentation in front of the class. As for its concrete formats and criteria, the authors recommend that students and teachers agree on them beforehand. In general, the admirers of this genre see both the structure and the topic of a portfolio, and its subdivision into specific sections, as emerging from a continuous process of discussions and consultations. It is gratifying that the cited guidelines stipulate that creativity and humor should be welcome during the compilation of a portfolio. We should note, however, that we have no evidence that this form of assessment enjoys widespread use in Russia today.

7 Exams and Oral Survey Tests

7.1 *Oral Survey Tests*

In the 1970s, oral survey tests began to be actively promoted in Russian schools. Within the framework of this system, after a class finished covering a large topic, all of the students in the class would be given survey tests (in general, orally). The formats of such tests were quite varied. Much depended on the topic being studied and the depth of understanding that the teacher wanted his students to achieve. When a survey test was given in oral form, teachers had to decide how long each test would last. Even if each student responded for only 5–10 minutes, even a flawlessly organized test could go on for 2–4 hours, if conducted by one teacher, because in the 1970s there could be as many as 40 students in a classroom. Teachers had different ways to solve this problem. Some teachers conducted oral tests mainly outside of class, scheduling appointments with students and talking to them quietly in their office. Since each teacher had no fewer than four classes, this occupied an inordinate amount of time. Many teachers would ask their colleagues to help them administer exams; with four teachers working together, the oral test could be completed relatively quickly, even within the span of one or two class periods. Such an approach could be productive, provided that three essential conditions were met: teachers had to have a genuine desire to help their colleagues; the demands within the school had to be consistent and uniform; and it had to be technically feasible to organize a lesson with the simultaneous participation of several teachers.

Another approach, which was widespread, was to have some students question and assess others. One of the founders of this approach was R. G. Khazankin, a mathematics teacher from the city of Beloretsk. He argued for the idea of "vertical pedagogy" (from upper to lower grades). Using this idea, Khazankin was able to achieve significant results (Khalamaizer, 1987). His methodology was never imposed on anyone, but many partly or wholly accepted its ideas. Within the framework of this approach, oral tests in grade 7 would be conducted by eighth graders, oral tests in grade 8 would be conducted by ninth graders, and so on. Finally, oral tests in grade 11 would be conducted by the school's graduates, who would return to administer the tests. Khazankin took part in and organized all of this. Naturally, such a system demanded not only a high level of organization, but also a high level of mathematical preparedness on the part of the students conducting the tests [of course, schools with an advanced course of study in mathematics had the best possible conditions for conducting oral tests in this respect as well (Karp, 1991)]. Far from all of Khazankin's followers were able to ensure that the level of the older students conducting a test was adequate for the aims and problems of the test that they were conducting.

Some educators went even further: within the framework of one and the same class, they would select several strong students, whose oral tests would be conducted by the teacher, and then these students "conducted oral tests" with their classmates. Such an approach was sufficiently dangerous, since the student conducting a test could turn out to be someone who mechanically learned material by rote and had no flexibility as a questioner. In such cases, an oral test could only be harmful. On the other hand, students from higher grades were not always able to "come down" to the level of a lower grade. Thus, for example, if a tenth grade student familiar with derivatives had to test an eighth grade student on the topic of the "quadratic function" and the eighth grader had to complete the square in order to find the vertex of a parabola, the tenth grader, instead of questioning the eighth grader, might begin explaining to the student how this could be done using derivatives. Another shortcoming of oral tests of this kind stemmed, naturally, from the distinctive character of the relationships between students in the same school and, even more so, within the same class. The objective of the oral test was also important. If, for example, the objective was to test how well ninth graders knew all the formulas of "school trigonometry" (without deriving them), i.e. if the assignment ruled out any ambiguities and was easy to grade, then students from a higher grade could handle the responsibility of administering it quite well. (In Soviet schools, and to a certain extent in Russian schools today as well, students were not permitted to use any reference books, notes, tables, calculators, etc., while taking tests. All formulas must be retained in memory.)

As an example, let us consider two different kinds of oral tests for a 10th-grade course on three-dimensional geometry. A *final* oral test, for instance, may confine itself to testing how well students have assimilated the course's basic theorems [the textbook by and Zvavich (2003) contains 35 such theorems]. For preparation, students are given a list of all of the course's theorems in the order in which they were studied. If in order to pass a given oral test a student must demonstrate the ability to formulate any formula on this list, make a diagram for it, write down what is given in the theorem and what must be proven, then it is perfectly feasible to let the oral test be conducted by 11th graders who have taken a similar course a year earlier. Such a test would be useful both to the 10th graders taking it and to the 11th graders administering it. On the other hand, if the purpose of the test is to test the students' ability to prove these theorems, to test how deeply they have grasped them, then it is much more advisable to let professional teachers conduct it, since such a test would demand of the individual administering it not only sound knowledge but also flexibility of thought.

The same reasoning may be applied to a *thematic* oral test: if the purpose of the test is essentially to check how well the students reproduce what they have learned (formulations, definitions, formulas), then it may be entrusted to 11th graders. However, if its aim is a deeper assessment, then the oral test ought to be conducted by the teacher, his or her colleagues or former graduates. As an example of the content of such a thematic oral test, consider the test conducted by one of the authors of this chapter for students at the beginning of 10th grade. This test included several topics studied at the beginning of the course in three-dimensional geometry — parallel projection, parallel planes, the angle between two planes, distance in space — as well as review sections on "quadrilaterals," a topic studied in eighth grade. To prepare for the test, students received assignment cards that would be used to conduct the test. Each assignment card contained two theoretical questions and two problems. Different approaches are possible here: the theoretical part, naturally, must be revealed to the students in advance, but the problems may be revealed either in advance (and thus used to test not how well the students solve problems, but how well they can explain their solutions) or on the test itself. The contents of one such card are reproduced below:

1. Parallel projection. The properties of parallel projection. Orthogonal projection and its properties.
2. The properties of a parallelogram.
3. Let the point K divide side $\overline{AA_1}$ of the cube $ABCDA_1B_1C_1D_1$ into two segments that stand in a relation of 2:1 beginning from A. Through the point K, trace a section of the cube that is parallel to the plane A_1C_1D, and construct an orthogonal projection of this section onto the face $ABCD$.
4. The bisector of angle A of the parallelogram $ABCD$ has divided its side $\overline{BC}$ in a relation of 3:7 beginning from B. Find the area

of the parallelogram if its perimeter is 1 m and one of its angles is twice as large as another.

To conclude this section, we should note that for convenience of discussion we are simplifying somewhat the variety of techniques and formats that are employed for conducting survey tests. In reality, such tests often make use of a combination of oral and written formats. For example, first a number of students (six or seven) may be asked to prove some basic theorems; while they are preparing their answers on the blackboard, the teacher may ask the other students to provide various definitions, give various kinds of examples, solve oral problems, and so forth (in other words, give them assignments that they can do quickly). After the presentation of the proofs and the discussion of the students' answers, which constitute the main part of the test, all of the students may be asked to prove some theorems or solve some problems in written form. Such a format, of course, is less effective than a full-blown oral test for assessing (and stimulating) the development of students' ability to express themselves orally, but it can nonetheless serve this purpose. It can also be organized with relative ease by a single teacher within the span of two class periods (90 minutes) and sometimes even one class period.

7.2 *Exams*

Final oral tests, referred to above, to some extent took the place of yearly final exams. At certain stages of its development, the Soviet (Russian) school system had no need for final tests, since every year ended with "transition exams," which students had to take to pass from one grade to the next, in many subjects and certainly in mathematics. At other stages, by contrast, all "transition exams" were abolished and only so-called graduation exams were left in place — at the end of basic school and secondary school, respectively. Below, we will talk about written graduation exams in mathematics for grades 9 and 11, which are conducted in a centralized manner. To begin with, however, we should like to point out that "transition exams" have today been left largely up to each individual school's discretion. Each school determines in which subjects to conduct exams and how to conduct them (while allegedly taking students' opinions into account). Sometimes such exams are written (as large final tests) and sometimes oral. Sometimes the problems on the exams are approved by, say, the district mathematics supervisor; sometimes they are not. No uniform system of requirements concerning such exams appears to exist in the country at the present time.

What has been said about "transition exams" applies also to oral graduation exams in geometry for grades 9 and 11 (more precisely, it used to apply to these exams, since recently the introduction of the USE has led to the elimination of other 11th-grade graduation exams). In recent years, these exams have been conducted because both schools and students have demanded them, and the assignment sets for them have been composed by teachers themselves (previously, their theoretical portions came from the Ministry of Education and only the problems on them were composed in the schools). Below is an example of one such assignment set (as many as 20–25 such assignment sets could be composed for one exam):

1. Parallel straight lines in space. The theorem about two straight lines that are parallel to a third straight line.
2. Distance in space. The geometric locations of points equidistant from two points, three points, two planes.
3. A problem on the topic "Vectors in space: the scalar product of vectors."

Students would know in advance the topic on which a problem would be given, but the problem itself would be revealed to them only on the exam. The exam would be conducted by a commission, which was usually chaired by the director or vice-principal and included the teacher who taught the class as well as one or even two other mathematics teachers.

Moving on to written graduation exams for grades 9 and 11, we can say that over the last quarter-century the principles of their composition have gone through radical transformations. Today, the USE, already mentioned numerous times above, has become the standard exam for 11th graders, but there is little cause to expect any kind of stability in this area, and it is possible that what is described here as contemporary and up-to-date will have changed by the time this book goes to press.

In the 1940s, the exam in algebra began to be composed in Moscow and then distributed throughout the country in sealed envelopes (Karp, 2007). These envelopes were supposed to be opened one hour before the start of the exam, but in practice their contents very often became known beforehand. On the other hand, the exam nonetheless inevitably possessed some kind of unpredictability — a student who really did not know the text of the exam in advance could encounter an unfamiliar formulation, a forgotten technique, and so on. The response to this came in the form of so-called "open" problem books, the first of which was a problem book for use in conducting exams in algebra at the end of basic schools, i.e. exams covering material from grades 1 and 8 and, later, grades 1 to 9 (MP RSFSR, 1985; latest edition, Chudovsky and Somova, 1995). The problem book included five sections, each of which contained 100 problems, given in two versions. During the school year, every student had to have a copy of the problem book in his or her possession. The exam envelope contained only five numbers, one from each section, and the problems with the corresponding numbers in the problem book were the problems that students had to solve on the exam (in two versions).

Such exams had their advantages and disadvantages. On occasion, during the second half-year of ninth grade, teachers would do nothing with their students except solving problems "from the problem book." Yet, it should be remarked that even this was not simple "drilling" in the strict sense of the word, since the problem book was sufficiently varied, and working with it, in our view, in one way or another facilitated both review and improvement in mathematical problem solving.

This remark also applies (albeit to varying degrees) to all of the later "open problem books" for exams in grades 9 and 11. The problem book mentioned above was replaced, in time, with a problem book by Kuznetsova *et al.* (2002), for use in standard public schools. The assignments for ninth-grade exams in classes with an advanced course of mathematics (gradually, the ninth-grade exam began to be conducted on two levels — for ordinary schools and for classes with an advanced course of mathematics) were sent in an envelope, "the old-fashioned way," but they consisted of six problems, most of which were drawn from the open problem book of Zvavich *et al.* (1994). In addition to these texts, the problem book of Shestakov *et al.* (2006) has been used and still remains in use for conducting exams in grade 9. Each of these problem books possesses its own methodological peculiarities, which cannot be discussed in detail here, especially since in recent times the so-called State Final Certification — a kind of analog to the USE in 11th grade — has come to be used with increasing frequency as the graduation exam for ninth grade. For this reason, we will limit ourselves to a relatively detailed analysis of the USE. But, first, let us say a few words about the way in which graduation exams in secondary schools used to be conducted in the past.

As has already been mentioned, until a certain time assignments for 11th grade, like assignments for ninth grade, would be sent in sealed envelopes from the Ministry of Education. As an example, a version of the problems from an exam in algebra and elementary calculus for an ordinary public school class is reproduced below (Zvavich, Shlyapochnik, and Kulagina, 2000, p. 15). The exam was meant to be completed in five hours, and students were given a grade of five for providing complete solutions to any five problems.

1. Find the intervals on which the function $y = 2x^3 + 6x^2 - 18x + 9$ is increasing and those on which it is decreasing.
2. Solve the equation $\sin 2x + \sqrt{3}\cos 2x = 0$.
3. Solve the inequality $\log_{\frac{2}{5}}(1 - x) \geq -1$.
4. For the function $f(x) = 2x - 6$, find the antiderivative whose graph intersects the x-axis at a point with the x coordinate 4.
5. Solve the following system of equations: $\begin{cases} 6^{2x} + 6^x \cdot y = 12 \\ y^2 + y \cdot 6^x = -8 \end{cases}$.
6. For which positive values a does the equation

$$(\log_3 a) \cdot x^2 - (2\log_3 a - 1) \cdot x + \log_3 a - 2 = 0$$

have a unique solution?

Even assuming that the solutions were written down in great detail and with great precision, five hours in the vast majority of cases was nonetheless an inordinate amount of time. Those students who could not solve these problems in three hours not only could not solve them on their own in five hours, but could likely never solve them at all. Nonetheless, it was not considered feasible to reduce the amount of time allotted for the exam.

Also, while during the Stalin and even the Khrushchev years the exam was indeed uniform in the full sense of the word, i.e. all students solved practically the same problems, later on, in the 1970s, special versions started to be prepared for schools with an advanced course of mathematics. Then, in the 1990s, versions began appearing for classes with a humanities specialization. Even later still, open problem books became part of the practice of conducting exams for ordinary public schools. Problems for the exams would be drawn mainly from the problem books of Dorofeev *et al.* (2002). Also, the problem books by Karp and Nekrasov (2001) and Shestakov (2006) were in use.

At the turn of the 21st century, the idea of introducing a Uniform State Exam surfaced in Russian education. By and large, the conception behind this exam has yet to be worked out, but what is clear is that it concerns not only exams in mathematics: at stake are the fundamental problems of the organization of education. Discussing the shortcomings of the previously existing system, critics pointed to the obvious unreliability of the information being received about the knowledge of secondary school graduates, on the one hand, and the evident and growing corruption of the system of college entrance exams on the other hand. College teachers, whose economic position declined significantly during the 1990s, for example, often undertook the preparation of students for entrance exams that they themselves would then administer, with understandable consequences. Indeed, students would take an exam in mathematics in school, and then go on to take a college entrance exam a month later, in keeping with the same official program. This was, for many, cause for perplexity.

In the early 1990s, attempts were made to unite the graduation and entrance exams, i.e. to treat the graduation exam conducted in school as a college entrance exam. The procedure that developed was not always perfect, if only because it relied on agreements between specific schools and specific colleges. As a consequence, students from one school would have their graduation exams counted also as entrance exams when applying to a given college, while students from a different school, which did not have such an agreement with that college, would not have the exact same graduation exam counted as an entrance exam for the exact same college, even if their scores on the exam were in fact higher. Perhaps a system of broader collective agreements, encompassing many colleges and all (or at least very many) schools, might have evolved over time, and Russia would have come to its own version of an exam that combines graduation and entrance exams, just as many countries in Western Europe have come to different versions of such an exam in the past. But the initiatives of isolated schools and colleges did not fit in with the "construction of the vertical of power" in the country, which began to be seen from a certain moment on as the supreme objective. As a result, all local experiments were curtailed, and an exam that was composed in one place for the entire country was established by command decision.

Rather quickly, however, it came to light that a number of the top colleges were permitted to enroll students in accordance with their own entrance procedures, after which it became difficult to claim that the USE was based on the principle of universal equity. The futility of hoping that the USE would be conducted in an absolutely honest fashion became clear after the publication of many scores and practically official admissions of existing infractions and "anomalously high and anomalously low scores" in various parts of the country (http://www.kremlin.ru/news/4701; http://www.echo.msk.ru/programs/assembly/595241-echo). However, the official position still maintains that it is the procedures for administering the exam that require improvement, that the leaders who condoned the falsifications need to be punished, that restrictions must be placed on the use of mobile phones which were used to dictate solutions to students, and so on — along with rebuking the children themselves and their parents, who did not do enough to prevent cheating on exams.

Debates about the USE in mathematics (see, for example, Abramov, 2009; Bolotov, 2005; Kuz'minov, 2002; Sharygin, 2002, as well as the website http://www.mccme.ru) have been and continue to be very heated, with the assignments themselves often being the first to come under fire. They are criticized, on the one hand, for lacking creativity and, on the other hand, for excessive difficulty, which is too forbidding even for children who were good students in ordinary schools (let alone those who attended schools with a humanities specialization) but who did not have additional lessons in mathematics. It is also argued that the assignments chosen for the exam give students a misleading impression of mathematics as an activity that is purely computational and on the whole lacking in substance. It seems likely, therefore, that the principles according to which these exams are composed will be changed in the near future, and likely changed more than once after that. At least, the so-called demo versions of the USE in mathematics for 2009 and 2010 (http://www1.ege.edu.ru/content/view/21/43/) are significantly different from one another.

We will confine ourselves here to discussing the 2009 exam (this was the first year that the USE in mathematics was taken by the whole country). This exam consisted of three parts and contained 26 problems.

Part 1 contains 13 problems (A1–A10 and B1–B3) at a basic level, drawing on material from the school course in mathematics. For each of the problems A1–A10, four possible answers are given, only one of which is correct. In doing these problems, students must indicate the number of the correct answer; in other words, these are multiple-choice questions. For problems B1–B3, students must give short answers.

Part 2 contains 10 more difficult problems (B4–B11, C1, C2) based on material from the school course in mathematics. For problems B4–B11, students must give short answers; for problems C1 and C2, they must write down solutions.

Part 3 contains the three most difficult problems, two in algebra (C3, C5) and one in geometry (C4). For these problems, students must write detailed and substantiated solutions.

The solutions to the problems in groups A and B were scanned and checked centrally (by a computer). The solutions to the problems in group C were checked locally by specially prepared groups of experts. The problems were given "raw" scores, which were then translated into final scores in such a way that a student who had answered every answer perfectly would receive a score of 100. The lowest boundary for a passing grade on the exam was determined after the exam had taken place. In 2009, a score of 21 turned out to be sufficient, which corresponded to four problems from group A. Of the students who took the exam in mathematics, 5.2% received a failing score. This result is alarming, to say the least — although it must also be said that the problems in group C, which are solved by a relatively large number of graduates, are quite difficult. One of them is reproduced below as an example (C4 from the demo version for 2009).

> A sphere whose center lies on the plane of the base ABC of the right pyramid $FABC$ is circumscribed around that pyramid. The point M lies on the side AB in such a way that $AM : MB = 1 : 3$. The point T lies on the straight line $\overline{AF}$ and is equidistant from the points M and B. The volume of the pyramid $TBCM$ is equal to $\frac{5}{64}$. Find the radius of the sphere circumscribed around the pyramid $FABC$.

8 Conclusion

The Russian system of assessment employs a variety of techniques, methods, and formats. It can also be implemented in different ways. In one class, the observer will marvel at the subtlety, precision, and cogency of the questions being posed, which simultaneously test and develop the students' understanding and knowledge. The observer will also see how obviously useful it is for the whole class to discuss what one student has said, and will take note of the quality of the teacher's comments — reasoned, even-handed, and well-received by each student and by the class as a whole. In another class, the same observer might behold a bleak scene of battle between the teacher, who effectively insults the students with his or her remarks, and the students, who not only no longer see the teacher, but do not see the subject itself anymore, which is indeed a subject represented by monotonous problems and hardly of any interest to anyone.

The effectiveness of this system has depended on the qualifications of the teacher. To be sure, the teacher's qualifications were by and large also formed in an environment in which value was placed on meaningful problems, in which such problems were produced and used often and in large quantities. To appreciate the accomplishments of the Russian system of assessment, one must first and foremost value its strictly mathematical side. The teacher of mathematics must know and love mathematics: this is something banally obvious, and yet, as everyone knows, it is by no means always and everywhere the case. It would, of course, be a plain falsehood to assert that this has always been the case in Russia. Nonetheless, it may be said with confidence that at a certain stage, Russia, for a whole range of reasons, possessed a comparatively large number of highly qualified teachers of mathematics, thanks to whom the Russian system of mathematics education in general, and the Russian system of assessment, in particular, were as effective as they were.

Will Russia preserve these traditions? The effects of political decisions in education are felt first in the sphere of assessment, and only subsequently everywhere else. In our view, for all the shortcomings of the previously established system of college entrance exams, this system raised the social status of the teacher, who could prepare students for a difficult exam. Both corruption, which renders real knowledge unimportant for passing college entrance exams, and the elimination of difficult exams have ruinous consequences for the teacher's status. On the other hand, those teachers also commanded respect who were capable of genuinely raising the level of all or almost all of even the weakest students to a three, a grade that was not all that easy to earn (to be sure, here, too, large-scale falsifications and the giving of threes in place of twos to fulfill the demands of the director of the school caused enormous harm). The situation is hardly improved by the fact that now, in order to get a three, it may be enough to copy four letters, i.e. four letters that stand for the right answers to four multiple-choice questions, from the screen of a mobile phone.

An educational system possesses its own kind of stability: it is difficult to reform it for the better, but it is also not so easy to destroy it. Whether Russian mathematics education will preserve the best aspects of its system of assessment, or whether only its traditional form will survive while its meaning and content vanish, or whether something totally new will appear in its place — only the future will tell.

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9

Extracurricular Work in Mathematics

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1 Introduction

This chapter is devoted to extracurricular work in mathematics in Russia. The traditions surrounding such work took shape over many decades. Without making any claim of giving a complete account of their history, we will say that in some measure these traditions date back to even before the Revolution of 1917. To be sure, the mathematics circles that appeared at that time for the most part brought together teachers who were interested in exchanging views on current issues in mathematics and its teaching methodology; yet some such circles attracted students as well (*Trudy*, 1913, p. 303). In the post-Revolution period, when traditional schools with their system of lessons and classes came under criticism, extracurricular work began to receive special attention. However, such forms of extracurricular work as subject field trips enjoyed the greatest popularity (Zaks, 1930).

During the radical restructuring of education in the 1930s, the orientation of extracurricular work changed as well. Research mathematicians began to play an active role in it. In 1934, an annual journal on mathematics education — *Matematicheskoye prosveschenie* — was founded which contained articles about interesting facts and problems of mathematics. In the same year, the first mathematics Olympiad in the Soviet Union was held in Leningrad, and in 1935 a similar Olympiad took place in Moscow. During these years, too, citywide mathematics circles for schoolchildren were formed in both cities. The most prominent mathematicians of the time actively participated in these circles, including A. N. Kolmogorov, B. N. Delone, L. A. Lyusternik, and others [Boltyansky and Yaglom's (1965) article remains a crucial source of information on the work of these mathematics circles].

Gradually, a body of literature for extracurricular classes developed. We might mention, for example, a series of books entitled *The Library of the Mathematics Circle* (Balk, 1959; Dynkin, 1952; Shkliarskii *et al.*, 1952, 1954, 1970, 1974, 1976; Yaglom, 1955, 1956), which reflected the activities of the Moscow circles, as well as the pamphlets in the series *Popular Lectures in Mathematics*, Ya. I. Perelman's books, and others.

The subsequent appearance of schools with an advanced course of study in mathematics and physics, and, more broadly, the popularity of and demand for mathematics and physics, facilitated the development and improvement of various forms of extracurricular work. All-Russia and All-USSR Olympiads became regular events. In the mid-1960s, I. Ya. Verebeychik, a teacher at school No. 30 in Leningrad, invented a new kind of mathematics competition, the "math battle," which quickly won popularity in the USSR. Among the other forms of extracurricular work that became popular, we might name mathematical contests and tournaments, mathematical theatrical evenings, field trips, elective classes, schools for young mathematicians, and others.

The years of social uplift at the end of the last century (1985–1990s) also witnessed a kind of explosion of extracurricular work in mathematics. In many cities, full-fledged organized systems of working with gifted students appeared at this time (Yarolslavl, Kostroma, Ivanovo, Kemerovo, Omsk, and others), and in major centers, where mathematics circles had already existed, their spectrum expanded.

Researchers outside Russia have some knowledge of Russia's experience with extracurricular work, if only from translated books (such as, Fomin *et al.*, 1996; Shkliarskii *et al.*, 1962). But most of this knowledge concerns approaches to working with the strongest students. Meanwhile, more modest, far less selective forms of extracurricular activity are equally of interest. We will describe them below, without aiming for a comprehensive account. The basic types of mathematical competitions are discussed in another chapter of this two-volume set (Saul and Fomin, 2010), so we will avoid focusing on competitions, except when it is indispensable for understanding the system of extracurricular work as a whole. We distinguish between the various forms of extracurricular work examined below based on the different segments of the school population at which they are aimed. At the same time, we recognize that any classification of real-life pedagogical activities will represent just one possible approach among many, and the various kinds of extracurricular activities examined below might be broken down into different categories — for example, based on the ages of the students at whom they are usually aimed (an aspect that we will also address, as far as possible).

2 Mass Forms of Extracurricular School Work

By using the word "mass" in the title of this section, we emphasize not the number of participants in the forms of extracurricular work discussed below — that number does not need to be very large — but the fact that they are aimed at *all* students, not just some group of students selected in advance, even if this group is very large. Extracurricular work begins in the classroom. This assertion will not seem paradoxical if one bears in mind that those who get involved in extracurricular work are interested students, and getting students interested must be done first in the classroom. Class work usually includes (or at least should include) problems of different levels of difficulty that might pique students' interest. Special supplementary sections in textbooks, which contain optional material addressed only to those who want to learn, serve the same purpose. A teacher may suggest that the students prepare a presentation on some topic, thus giving them an opportunity to become acquainted with additional literature now outside of class. Yet, special extracurricular forms of work that are specifically addressed to all students are also useful. We describe several of them below.

2.1 *Mathematical Wall Newspapers*

The usefulness of mathematical wall newspapers has always been emphasized in the Russian (Soviet) methodological literature (for example, Stepanov, 1991). Indeed, all students will look at a wall newspaper inside a classroom or next to its entrance, and many of them will likely read some part of it attentively. The content of such a newspaper may vary, but it is clear that it must, on the one hand, attract attention and, on the other hand, be sufficiently easy to read — standing in front of a newspaper for hours is hardly feasible. Consequently, such newspapers have often contained stories about various outstanding mathematicians (with their portraits and other interesting pictures and sufficiently interesting historical stories, none of which are difficult to find). They have also included various entertaining problems (again, if possible, with pictures). Such newspapers may also contain various kinds of practical information — announcements about mathematics circles, the results of various class or school competitions, and so on. Problems from written problem-solving contests, which will be discussed below, may also appear in wall newspapers.

One should not expect, of course, that reading a wall newspaper in itself will steer students toward doing mathematics on their own The goal here is different: to attract students' attention and perhaps to inform them about other extracurricular activities being offered. At the same time, if a wall newspaper is published regularly, then it usually acquires an editorial board: certain students who systematically choose material for it and gain a considerably deeper acquaintance with such material in the process. This, of course, concerns only a small group of students.

2.2 *Mathematical Theatrical Evenings and Oral Mathematics Journals*

The activities discussed in this section can go by different names, but all of them involve asking students to participate (on the stage or as members of an audience) in a theatrical presentation. Most often, such forms of extracurricular work are used with students of grades 5–7: their purpose is not so much to teach students mathematics, and maybe not even to get students interested in mathematics, as to demonstrate the "human face" of mathematics.

The script of a mathematical theatrical evening may include, for example, the following sections (Falke, 2005):

- Presentations about mathematics delivered from the point of view of other school subjects (mathematics and Russian literature, mathematics and physics, etc.);
- A parade of the "components of mathematical beauty" (students who represent symmetry, proportion, periodicity, etc., tell about these concepts, offering examples);
- A reading of poems about mathematics;
- A story about some great mathematician;
- Scenes with mathematical content, performed by the students;
- Mathematical questions for the audience; and so on.

Naturally, for such a theatrical evening to be a success, it is necessary to write a good script, do a good deal of rehearsing, possibly prepare costumes, and so on. None of these activities are usually considered mathematical; nonetheless, it may be expected that the teacher who has undertaken to supervise them will endow the students with a positive attitude toward studying mathematics.

Stepanov (1991) described an "oral journal" for seventh graders in a school, the purpose of which was to publicize a new mathematical elective being offered: "The pages of the journal were given to a ninth grader ("Sufficient Conditions for Divisibility"), an eighth grader ("How People Counted in Ancient Russia"), a mathematics teacher ("Symmetry in Mathematics and Around Us"), and the economist parent of one of the students" (p. 6). After the conclusion of the journal, the program of the new elective was displayed, and students had a chance to sign up for the course.

Actual mathematical activity — problem solving — is usually not a large part of such theatrical evenings. The "questions for the audience," mentioned above, may be completely elementary: "Can the product of two integers be equal to one of them?", "Is the difference of two positive integers always a positive integer?" (Falke, 2005, p. 28). However, a theatrical evening may also include a small competition in which students solve more difficult problems.

As an example of such an entertaining and comparatively easy problem, consider a question given at the so-called "mathematics festival" in Moscow, which constitutes a special Olympiad for grades 6 and 7:

> A kilogram of beef with bones costs 78 rubles, a kilogram of beef without bones costs 90 rubles, and a kilogram of bones costs 15 rubles. How many grams of bones are there in a kilogram of beef? (Yaschenko, 2005, p. 10)

To solve this problem, it is enough to note that a whole kilogram of beef costs 75 rubles more than a kilogram of bones, and 12 rubles more than a kilogram of beef with bones. Consequently, the share of bones in a kilogram of beef with bones is $\frac{12}{75} = \frac{4}{25}$. From this, it is clear that a kilogram of beef with bones contains 160 grams of bones.

2.3 *Mathematical Tournaments*

In contrast with mathematical theatrical evenings, mathematical tournaments are entirely devoted to competitive activities, which may be conducted, for example, in a class and consist of answering engaging questions. The questions for such contests may be prepared by the teacher or by the students themselves. A mathematical tournament may involve the participation of the whole class, for example, divided into two or three teams. Verzilova (2007) offered a detailed description of such an event for sixth graders, which we reproduce in abridged form:

> The program for the event is put on display one week before the event takes place. The teams are given homework assignments (see below). A panel of judges consisting of students from higher grades

is set up. After an opening statement from the master of ceremonies, the competitions begin. They include the following:

Auction
A set of triangles is put up for auction. The teams take turns to state facts about the topic "Angles" (the homework assignment included a review of this topic). The last team that can state a fact about angles wins the set of triangles.

Experiments with a sheet of paper
The teams have several sheets of paper, some square and some irregularly shaped. They are given the following assignments:

1. Fold a sheet of paper to obtain a right angle.
2. Fold a sheet of paper to obtain a 45° and a 135° angle.
3. Fold a sheet of paper to obtain a rectangle.
4. Take a square, fold it along its diagonals, and cut it along the lines of the folds. Using the obtained shapes, assemble: (a) two squares; (b) a rectangle; (c) a triangle; (d) a quadrilateral that is not a rectangle; (e) a hexagon.

Eye test
Several different angles, made of transparent colored film, are projected onto a screen. The members of all of the teams are asked to estimate their degree measures and write them down on a sheet of paper. Then, using a transparent protractor, the angles are measured and all of the participants write down the correct results next to their guesses. The sheets of paper are then submitted to the judges for determining which team has the most right answers.

Scientific fairy tales
Each team is asked to read two fairy tales, which they have composed in advance as part of their homework assignment. The remaining fairy tales are given to the judges to determine the winners of the homework competition. Here is an example of a fairy tale composed for such an event:

Adjacent angles

Once upon a time, two angles lived in the same house. They did not look like each other, because one was obtuse and the other acute. Their names were angle AOB and angle COB. It was impossible to separate them,

> *since they had one side in common and their other two sides formed a straight line. The angle brothers got along very well with each other, never leaving each other's side. Most of all, they wanted to invent a name for their house. They thought for a long time and finally decided to name their house after themselves: "adjacent angles."*

Several other contests follow. The mathematics festival concludes with the judges determining the winners and handing out awards.

2.4 *Written Problem-Solving Contests*

Optional problem-solving contests may be conducted in a class (or a school). Of course, far from all students take part in such contests (let alone successfully solve all problems), but all students in a class (or a school) are invited to participate in them, and that is why we discuss this activity in this section. Such contests are useful both in themselves and as a means of drawing students into a mathematics circle (where, for example, they will be told the solutions). Contest problems may be given in wall newspapers, as already mentioned. They may be given one or two at a time, for example as weekly assignments. Whatever the case, they are usually given for a sufficiently long period of time and thus presuppose that the participants have attained a certain degree of maturity and responsibility. It must be pointed out, too, that students are almost always unaccustomed to turning in work in which not all problems have been solved (and usually even the winners do not solve all of the problems). Consequently, it is very important to explain to potential participants that they are in no way expected to solve all of the problems.

In general, the psychological aspects of such contests usually require a fair amount of attention. If a contest turns out to be too easy, then the stronger students will not want to solve and hand in the problems; if it turns out to be too difficult, then, on the contrary, no one except a very small number of students will decide to participate in it. Consequently, a certain balance is necessary. Likewise necessary is a balance between comparatively traditional, "school-style" problems and problems with interesting but unfamiliar formulations. The following problems, for example, were used in a contest for seventh graders (Karp, 1992, pp. 10–11):

1. Solve the equation $|x - 1| + |x + 2| = 4$. (This is a typical, "difficult" school-style problem. The students have already analyzed absolute value problems, but even a single absolute value in seventh grade made a problem difficult, while this problem contains two of them. On the other hand, there is nothing particularly unexpected here: carefully following the algorithm for removing the absolute value sign, for example, will lead to the right solution.)
2. Two squares, with sides 12 cm and 15 cm, overlap. Removing the common part from each of the squares, we obtain two regions. What is the difference of their areas equal to? (In this case, for a person with a certain mathematical background, everything is very straightforward: regardless of the area of overlap of the two squares, the difference of the areas of the obtained regions is equal to the difference of the areas of the squares. But far from all students are capable of justifying this argument clearly and correctly.)
3. A scary dragon has 19 heads. A brave knight has invented an instrument that can chop off exactly 12, 14, 21, or 340 heads at once, but after this the dragon grows 33, 1988, 0, or 4 new heads, respectively. Once all of the heads have been chopped off, no new heads will grow. Will the knight be able to slay the dragon? (This is a typical, although not difficult, Olympiad-style problem: the students must note that the number of heads always changes by a multiple of three, and thus there is no way to pass from 19 — a number not divisible by 3 — to 0.)

3 School Mathematics Circles and Electives

In this section, we describe school mathematics circles and electives. We should emphasize that we will be discussing specifically mathematics circles formed within one school, usually an ordinary school (a different section will be a more natural place for a discussion on mathematics circles in specialized schools with an advanced course in mathematics). Of course, only a fraction of the students at a school become involved in such circles; nonetheless, these circles have never been, are not, and cannot be especially selective: the problem that they set before themselves is not to prepare future winners of All-Russia or even municipal Olympiads, but rather to facilitate general mathematical development.

To give a more complete picture, however, we should point out that school Olympiads are by no means limited to such high-level competitions as the just-mentioned municipal or All-Russia Olympiads. There is also a broad-based, district-level round, success in which is generally encouraged. The quite numerous forms of accountability that have existed and continue to exist in schools have included providing reports about work not only with the "bottom" of the student body — about the so-called struggle against academic failure — but also with the "top" of the student body, for example about students' achievements in Olympiads. Predictably, this has led to contradictory results: on the one hand, teachers have often found comparisons between their activities in this respect unfair (not without reason) — obviously, students at more selective schools show better results than students at ordinary schools, and it is hardly possible to blame teachers at ordinary schools for this; on the other hand, this kind of official attention has nonetheless motivated teachers (even if not all of them) to devote more thought to working with stronger students.

The district-level rounds of Olympiads include problems which, even though they are not, generally speaking, especially difficult, nonetheless differ substantially from the problems ordinarily solved in the classroom. An as example, consider the following problem from a district-level round for sixth graders:

> Each of three players writes down 100 words, after which their lists are compared. If the same word appears on at least two lists, then it is crossed out from all the lists. Is it possible that, by the end, the first player's list will have 54 words left, the second player's 75 words, and the third player's 80 words? (Berlov *et al.*, 1998, p. 15)
>
> The solution of the problem is based on a simple line of reasoning. If the described outcome were possible, then the first player would have 46 words crossed out, while the second player and third player

would have 25 and 20 words crossed out, respectively. But $20 + 25$ is less than 46. Therefore, not all of the 46 words crossed out on the first player's list could have been on the other players' lists.

Although no special prior knowledge is required to solve such a problem, those who have had experience with solving problems that are not "school-style problems" have found themselves in a better position at Olympiads. Teachers are advised to conduct so-called school-level rounds (Olympiads within a school), which are supposed to, on the one hand, prepare students for the district-level Olympiad and, on the other hand, select those who will be sent to the district-level round. In practice, such school-level rounds are very often skipped, and the problems suggested for school-level Olympiads are used in some other capacity (for example, put on display, along with their solutions, to allow students to become acquainted with them) or not used at all, and teachers themselves decide whom to send to the district-level round (the selection is usually not rigid, however, and students who wish to take part in the district-level round can usually do so). We should emphasize once more, however, that a systematic mathematics circle can help students to prepare for an Olympiad.

Officially, the differences between mathematics circles and electives have been (and remain) quite substantial. Generally speaking, students have the right to choose which electives they wish to attend, but once this choice is made they are required to attend the elective which they have selected; by contrast, participation in a mathematics circle remains voluntary at every stage (to be sure, a teacher can, in certain situations, prohibit students who skip mathematics circle meetings too often from attending at all). The wages received by teachers for teaching mathematics circles and electives are somewhat different as well (it should be noted that teachers have sometimes taught mathematics circles with no compensation at all). Nonetheless, it is not always possible to make a sharp distinction between the programs of mathematics circles and electives. A topic that has officially been included in the program of electives may become the basis for a mathematics circle. The more "adult" word "elective" is heard more frequently in the higher grades; in grades 5–6, only the term "mathematics circle" is used.

3.1 *Mathematics Circles in Grades 5–6*

We will draw on Sheinina and Solovieva's manual (2005) to provide a rough description of the work of such a mathematics circle. It contains material for 30 sessions (1.5–2 hours each) and, as the authors remark in their annotation:

> ...it was written with the aim of helping the teacher of a school mathematics circle to conduct systematic sessions (at least two per month) [whose purpose is] to interest the students, supplementing educational material with facts about mathematics and mathematicians, to improve students' mental arithmetic skills, to develop their basic mathematical and logical reasoning skills, to expand their horizons, and above all to awaken their interest in studying one of the basic sciences [namely, mathematics].

As an example, consider one mathematics circle session outlined in the book (No. 14). The session consists of several sections, material for which is provided. At first, students are given various puzzles, among which, for example, is the following problem:

> Express the number 1000 by linking 13 fives in arithmetic operations (for example, $5 \cdot 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5$).

Next come several "fun questions":

- Five apples must be divided among five children so that one apple remains in the basket.
- Two fathers and two sons shot three rabbits, one each. How is this possible?
- How many eggs can be eaten on an empty stomach?

(The answers are, respectively, that one child must be given the basket with one of the apples inside it; that the rabbits were shot by a grandfather, a father, and a son; and that only one egg can be eaten on an empty stomach.)

Next, the students are given a brief biography of Newton. This is followed by a section called "Solving Olympiad problems." Here, students are asked to use trial and error to find the solutions to the equation $2y = y^2$, to solve a rather long word problem, and to say whether a boy has 7 identical coins if he has a total of 25 coins in denominations of 1, 5, 10, and 50 kopecks. The session concludes with a poetry page: the students read a poem about the Pythagorean theorem, and so on.

The methodology of conducting a mathematics circle session is not discussed in the manual, but it may be assumed that, for example, the biographical vignette is presented by the teacher or by a specially prepared student. The poetry page is likely approached in a similar fashion.

The examples given above show that the work of a mathematics circle can hardly be characterized as intensively mathematical: what we see, rather, is work focused on the students' general development. Nonetheless, mathematics circles play an obvious role in instilling in students a positive attitude toward problem solving and studying mathematics in general.

3.2 *Mathematics Circles and Electives in Grades 7–9*

In working with students from grades 7–9, less attention is devoted to the "entertaining" side of things, naturally, than in working with students from grades 5–6. The program of study becomes more systematic. Nikolskaya's manual (1991), published in Soviet times, recommended the following program of study for elective courses in these grades:

Grade 7

- Number systems
- Prime and composite numbers
- Geometric constructions
- Remarkable points in a triangle

Grade 8

- Number sets
- The coordinate method
- Elementary mathematical logic
- Geometric transformations of the plane

Grade 9

- Functions and graphs
- Equations, inequalities, systems of equations and inequalities
- Remarkable theorems and facts of geometry
- The logical structure of geometry

In other words, such a program involves expanded study of the existing school program (indeed, since the collapse of the Soviet Union, with schools acquiring greater opportunities, such a program or one similar to it has sometimes simply been added to the ordinary school curriculum, with the classes that study this expanded curriculum being labeled as classes with an advanced course in mathematics).

The manual by Gusev *et al.* (1984), published even earlier, suggested a number of topics for extracurricular work in grades 7–9 (6–8 in the system that existed at the time), which largely resembled the topics found in mathematics Olympiads. Among them, for example, were such sections as "Graphs," "The Arithmetic of Remainders," "How to Play in Order Not to Lose," and "Pigeonhole Principle." For each topic (which usually occupied several class sessions), the manual offered problem sets and provided specific methodological recommendations, such as suggesting various general theoretical facts that the teacher could convey to the students in one way or another, or describing various kinds of activities that the teacher might organize.

In fact, during those years as well as later, in school electives and mathematics circle sessions, students usually solved problems of a heightened level of difficulty. For the most part, these problems were based on material from the ordinary school curriculum, but they could also include problems that drew on traditional Olympiad-style topics.

For example, problems involving absolute value or problems that required students to construct nonstandard graphs [for instance, construct the graph of the equation $y + |y| = x$ (Kostrikina, 1991, p. 46)] have always been very popular. The same is true of problems on solving equations and inequalities, as well as word problems based on equations and inequalities. Also represented were identity transformations, problems on progressions, and trigonometry. Kostrikina's (1991) problem book, cited above, contains problems of a heightened level of difficulty in practically all sections of the course in algebra. Consider several more examples of problems from this text:

- Find a two-digit number that is four times greater than the sum of its digits. (p. 43)
- What is greater, $\frac{10^{10}+1}{10^{11}+1}$ or $\frac{10^{11}+1}{10^{12}+1}$? (p. 48)
- Simplify the expression $\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}}$, if $1 \leq x \leq 2$. (p. 102)
- For what value of a is the sum of the squares of the roots of the equation $x^2 + (a-1)x - 2a = 0$ equal to 9? (p. 108)
- Prove that the greatest value of the expression $\sin x + \sqrt{2}\cos x$ is equal to $\sqrt{3}$. (p. 180)

The first two of these problems are recommended for grade 7, the next two for grade 8, and the last one for grade 9. As can be seen, these and similar problems placed rather high demands on students' technical skills, but the reasoning skills required to solve them were also quite high (of course, students were also given simpler problems to solve in mathematics circles and electives — the examples above were chosen to illustrate the types of problems offered).

There is a considerable amount of material in geometry for school extracurricular work. The curriculum for grades 7–9 contains a sufficiently complete and deductive exposition of Euclidean plane geometry; this material may be used as a foundation for posing problems that are quite varied in character. Indeed, school textbooks themselves usually provide considerably more material than can be studied and solved in class. Among the supplementary manuals, we should mention the popular and frequently reprinted problem book by Ziv (1995), intended for use in ordinary classes, but containing more difficult problems recommended for mathematics circles. Again, since lack of space prevents us from describing these problems in any detail, we will confine ourselves to a single example:

> A point D is selected inside a triangle ABC. Given that
> $m\angle BCD + m\angle BAD > m\angle DAC$,
> prove that $AC > DC$. (Ziv, 1995, p. 59)

The solution of this problem, which is assigned to seventh graders, is based on the fact that the longest side of a triangle lies opposite the largest angle and on the properties of a triangle's exterior angle. However, to arrive at this solution, the students must possess a certain perspicacity and, above all, a comparatively high level of reasoning skills. Solving geometric problems as part of extracurricular work (and usually in classes as well) practically always involves carrying out proofs of one kind or another.

Evstafieva and Karp's (2006) manual gives an idea of what kind of typical Olympiad-style material might be studied in mathematics circles. This collection of problems, intended mainly for working with ordinary seventh graders in ordinary classes, contains a section entitled "Material for a Mathematics Circle." This section has five parts:

- Divisibility and remainders
- Equations
- Pigeonhole principle
- Invariants
- Graphs

As can be seen, the topics are quite traditional for mathematics circles of even higher levels (Fomin *et al.*, 1996). But here the assignments are limited to relatively easy problems, the number of which, however, is relatively large and which are organized in such a way that, after analyzing one problem, the students can solve several others in an almost analogous fashion. For example, the following three problems appear in a row:

- The numbers $1, 2, 3, 4, \ldots, 2005$ are written on the blackboard. During each turn, a player can erase any two numbers x and y and write down a new number $x + y$ in their place. In the end, one number is left on the board. Can this number be 12,957?
- The numbers $1, 2, 3, 4, \ldots, 2005$ are written on the blackboard. During each turn, a player can erase any two numbers x and y and write down a new number xy in their place. In the end, one number is left on the board. Can this number be 18,976?
- The numbers $1, 2, 3, 4, \ldots, 2005$ are written on the blackboard. During each turn, a player can erase any three numbers x, y, and z, and write down two new numbers $\frac{2x+y-x}{3}$ and $\frac{x+2y+4z}{3}$ in their

> place. In the end, two numbers are left on the board. Can these numbers be 12,051 and 13,566? (p. 150)

A problem of the same type as the first of these problems (although slightly more difficult) also appears in the aforementioned manual by Fomin *et al.* (1996). The solution to the problem above is very simple: the sum of the numbers on the board does not change after the given operation, and consequently the number left on the board at the end must be equal to the sum of all the numbers that were on the board at the beginning, which is obviously not the case if the last number is 12,957 (note that the problem is posed in such a way that this answer is obvious in the full sense of the word — it is not necessary to find this sum). But in the problem book that is aimed at a more selective audience, the very next "similar" problem is far more difficult, whereas in the case above it is relatively easy for the students to determine what remains invariant in the subsequent problems; they might be asked to invent an analogous problem on their own, and so on. In other words, the goal is not so much to solve increasingly difficult problems by using a strategy that has been learned as to become familiar with this strategy itself — in this instance, with the concept of invariants.

Thus, the topics studied in mathematics circles are often mixed, including some amount of Olympiad-style problems and typical difficult school-style problems.

3.3 *Mathematics Circles and Electives in Grades 10–11*

Although we lack any firm statistical evidence, we would nonetheless argue that mathematics circles and electives in higher grades of ordinary schools are devoted mainly to solving difficult school-style problems (naturally, there are exceptions). Those students of ordinary schools who wish to enter colleges with more selective programs in mathematics, search for opportunities to prepare better for exams (traditionally, each college had its own entrance exams; now they have been replaced by a standard exam for the entire country — the USE). According to our observations, various manuals for preparing for the USE have become an important source for such preparation literally in the last few years (such as Semenov, 2008). Consider the following problem as an example:

> Among all the integers that do not constitute solutions to the inequality $(10^{4x-9} - 1)(3^{5x-21} - 1) \geq 0$, find the integer that is the least distance from the set of solutions to this inequality. (Semenov, 2008, p. 69)

The solution of this somewhat artificial, although not difficult, problem requires solving an exponential inequality, defining the integers that do not belong to the corresponding set, grasping the very notion of the distance from a number to a set, and finally comparing numbers (fractions). Clearly, such an exercise requires a good bit of time.

Examining the content of school electives in higher grades, we must mention two books, Sharygin (1989) and Sharygin and Golubev (1991), which brought together many difficult problems from the entire range of school mathematics, thus making these problems accessible to teachers of school electives. The problems in these books were often organized and classified in terms of the methods used for solving them. As a result, the books were not simple. But they came to exert an evident influence on many subsequent publications. Consider the following example of a relatively easy problem from these texts:

> Given a right triangle ABC with legs $AC = 3$ and $BC = 4$ and two points M and K, such that $MK = 8$, $AM = 1$, and $BK = 2$, find the area of triangle CMK. (Sharygin, 1989, p. 167)

This problem is offered as an illustration of the notion that in a geometric problem it is important to identify the distinctive features of the figure that is given, and, in particular, the role of the numbers given. Indeed, once we start to draw the figure, we notice that

$$MK - AM - BK = 5 = BC.$$

This means that the points M and K lie on the straight line $\overleftrightarrow{AB}$. Since it is given that $MK = 8$, all that remains to be done is to find the length of the altitude from the vertex C to the straight line $\overleftrightarrow{AB}$, which is not difficult at all. The length of the altitude equals $\frac{12}{5}$, and the area we seek equals $\frac{48}{5}$.

4 On Various Forms of Distance Learning

In this section, we will begin to discuss forms of extracurricular work that take place outside specific schools (although, of course, the role of the teacher and the school in providing information about them to the students and in offering subsequent support is very important). The first activity of this kind that should probably be mentioned is independent reading.

Above, we referred to many books published specifically for schoolchildren interested in mathematics. Both in the USSR and, later on, in Russia, numerous collections of difficult problems have been published and republished, along with comparatively short and accessible presentations of various mathematical theories. In particular, we would single out books from the series "The Little *Kvant* Library," as well as the already-mentioned pamphlets from the series *Popular Lectures in Mathematics*. Books with a more explicit and closer connection with the school course in mathematics, which are intended for an audience of many thousands or perhaps even many millions, have been and continue to be published as well. Among them, we would single out books published under the general title *Supplemental Pages for the Textbook*.

Depman and Vilenkin's book (1989 and other editions), addressed to fifth and sixth graders, contains, for example, the following sections:

- How people learned to count
- The development of arithmetic and algebra
- From the science of numbers
- Mathematical games
- Mathematics and secret codes
- Stories about geometry
- Mathematics and the peoples of our homeland
- How measurements were made in antiquity, etc.

This book is, to be precise, not a textbook. Students can (and will want to) read it at home on their own. It is written in a colloquial style and contains many historical and entertaining facts, but also includes a considerable number of problems and stories about various areas of mathematics.

Supplemental Pages for the Algebra Textbook (Pichurin, 1999), a book addressed to students in grades 7–9, is written in a somewhat drier style, but has the same objective: to give an accessible and entertaining account of topics which comparatively strong students could have been told about in class, but which inevitably remain beyond the bounds of the school course in mathematics. The text includes stories about the evolution of algebra and several of its sections (for example, Diophantine equations or continued fractions), and generally attempts to identify key mathematical ideas and stages in the development of mathematics (for example, a section entitled "Turning Point in Mathematics" tells about Descartes's contribution and the appearance of the concept of variables).

Other books in the series were meant to accompany other parts of the mathematics curriculum, such as, *Supplemental Pages for the Geometry Textbook* (Semenov, 1999) and *Supplemental Pages for the Mathematics Textbook* for grades 10–11 (Vilenkin *et al.*, 1996). The purpose of these and other books was to support independent reading and self-education by students. Thus, Pichurin (1999) concluded his book with a section entitled "Reading Is the Best Way to Learn," in which he listed various books that interested students could use to continue their mathematical education.

It might be noted, however, that while the aforementioned book by Pichurin was published in 1990 in an edition of 500,000 copies (in Russia, the size of the edition is indicated in the book), in 1998 it was reissued in an edition of only 10,000 copies; the whole system of book publishing had undergone a radical transformation. Nonetheless, independent reading remains an extremely important way for many thousands of students to become more closely acquainted with mathematics. Moreover, the limited availability of printed books is partly compensated for by the Internet; for example, the website of the Moscow Center for Continuous Mathematical Education contains quite a decent mathematics library.

Yet, no matter how significant independent reading may be, a student often cannot get by without guidance from a teacher. Sometimes, teachers can and want to offer such help, and that is all to the good; but even in the absence of such support in school, students can acquire help by taking classes at a mathematics correspondence school. Mathematics correspondence schools were originally created in the 1960s under the supervision of one of the greatest Russian mathematicians, Israel Gelfand. Together with his collaborators, Gelfand personally developed the programs for these classes and wrote textbooks for students. The idea was not to allow students who lived in regions that were far from the academic centers of the Soviet Union to slip through the cracks. The work of such schools is based on a simple principle. Students who enroll in them receive pamphlets in the mail with expositions of various areas of mathematics, examples of problems with solutions, and problems to solve on their own. The students solve these problems and send them back to schools, where they are usually checked and graded by students from the universities under whose aegis the correspondence schools operate; after which, the graded homework assignments are sent back to the students. Gradually, a framework developed in which not just individual students could enroll as students in correspondence schools, but entire classes or groups of students could do so as well (as a "collective student"). Within such a framework, teachers at ordinary schools could inform and organize their students, and at the same time learn together with them and continue their own education.

Since it is impossible for us to cover all details here, we can do no more than simply mention correspondence mathematics Olympiads (Vasiliev *et al.*, 1986), which became an important form of Olympiad activity — and quite distinctive in character, since problems that were assigned for solving over an extended period of time at home needed to be somewhat different from problems used in ordinary Olympiads, which had to be solved on the spot. We will, however, say a few words about the "ordinary" assignments given in correspondence schools. As an example, we will use one of the assignments of the Petersburg Correspondence School (centers of correspondence work also sprung up outside of Moscow).

The pamphlet *Problems in Algebra and Calculus* (Ivanov, 1995) is mainly devoted to solving problems, whose formulations resemble ordinary school-style problems, by using ideas from calculus and combining these ideas with standard ideas from the school curriculum.

The exposition begins with an analysis of several problems and a discussion on the intermediate value theorem, which is employed in their solutions. Among the variety of problems analyzed is the following: "For what values of the parameter a does the equation

$$\sqrt{2-x}+\sqrt{2+x}=x^2+a$$

have a solution?" (p. 2). The solution becomes obvious if one uses the derivative to sketch the graph of the function $y=\sqrt{2-x}+\sqrt{2+x}$ and determines its maximum and minimum. Another section of the pamphlet is devoted to function composition and the concept of the inverse function. Here, a certain theory is presented (again in the form of solutions to several problems), and then different ways of utilizing it are demonstrated.

Based on the analyzed material, several problems are posed. Among them are the following:

- Prove that the equation $\sin x = 2x + 1$ has a single solution.
- How many solutions, depending on the value of a, does the following equation have $\sqrt{x^2-4}=a-x^2$?
- Is it true that function f is invertible if the function $g(x) = f(x^3)$ is invertible?

The pamphlet contains 24 analyzed examples and 40 unsolved problems. Its material forms the content for two gradable assignments (15 problems each). To receive the highest grade (5), students must solve no fewer than 11 problems in each assignment, and to receive a satisfactory grade (3), they must solve no fewer than 7 problems.

The content of the pamphlet described here has a pretty close resemblance to the curriculum of so-called schools with an advanced course in mathematics. The topics in the pamphlets for correspondence schools, however, have varied: some pamphlets have dealt with traditional topics studied in ordinary schools, such as linear and piecewise linear functions, while others have addressed topics traditionally found in mathematics Olympiads (for example, the same invariants) or still other, untraditional subjects [the very title of one of the sections in Vasiliev *et al.* (1986) is noteworthy in this respect: "Unusual Examples and Constructions"].

5 Selective Forms of Working with Students

In large cities, various venues for extracurricular work with students appear, which bring together students not only from one school but from many schools (or, even if in some cases such venues are based in a single school, this is a specialized school with an advanced course in mathematics, which in turns selects children from the whole city). This section addresses the work that takes place at such venues. We will say at once, however, that our description will be relatively brief; more detailed information about many of the issues raised below may be found in Fomin *et al.* (1998), already mentioned above.

The word "selection" itself requires clarification. Even when we describe a mathematics circle that is highly selective, we should not necessarily assume that students must pass some exam to join the circle. Mathematics circles are formed in various ways: sometimes, indeed, by means of special invitations from the instructor, which are in turn based on the results of an Olympiad — only the winners are invited; but sometimes mathematics circles, when they are being formed, are open to all interested students. It is another matter that usually a process of natural selection occurs, as it were, when some of the students stop attending the sessions of the circle because they become interested in something else (and it must be borne in mind that once a group of mathematics circle attendees takes shape, it endures for several years — ideally until the students graduate from school). Other students sometimes discover that they are unable to handle the workload; there may even arise situations in which the instructor virtually expels a student from the class for some reason.

In general, it must be said that the situation in a mathematics circle depends to a very great degree on the instructor. For this reason, we must say a few words about where such instructors come from. There are no special programs that prepare teachers for mathematics circles, although proposals to create such programs have already been made in the professional community. Initially, even before World War II, citywide mathematics circles were created by professors, graduate students, and undergraduate university students. David Shkliarskii, a talented young mathematician who perished during the war, was an outstanding, although in some respects typical, representative of these early years of mathematics circles. It was Shkliarskii who transformed the structure of Moscow's mathematics circles, replacing the previously prevalent practice of students delivering reports with the systematic solving of difficult problems (Boltyansky and Yaglom, 1965). The new system was largely invented by him, and the problems for the mathematics circles were created and selected by him along with other young (or even not-so-young) mathematicians (who, naturally, were well aware of the relevant work that had been done before them in the field of mathematics education). Gradually, however, new generations grew up, consisting of individuals who had themselves been raised within the framework of the system of mathematics circles. Indeed, a kind of systematic mathematics-circle education developed, an education that was quite narrowly specialized, so that former participants in mathematics circles were sometimes accused of being clannish and cut off from broader interests — not just in their lives, but even within mathematics itself. At the same time, because many individuals who had gone through mathematics circles were also winners of highly prestigious Olympiads, participation in a circle became a prestigious matter — and being the teacher of a circle even more so.

Again, at a certain stage, instruction for mathematics circles was supported by the state, even if not financially. For young university students and graduate students, so-called "public service" was considered indispensable. Being the instructor of a mathematics circle was seen as a form of public service. Subsequently, with the collapse of the USSR and the disappearance, for example, of the Komsomol organization, the situation changed but the tradition remained intact. If one looks at the list of authors who wrote the problems for an Olympiad, it is usually not hard to notice that practically all of them had themselves been winners of prior Olympiads. The same individuals usually become instructors in mathematics circles.

Here, an additional clarification is again necessary. The number of Olympiad winners is not that great, and yet hardly all of them go on to become involved with mathematics circles and Olympiads (and certainly not all of them remain involved with them three or four years after graduating from high school). For example, in St. Petersburg, where the citywide Olympiad is oral and thus must be conducted by highly qualified examiners, special efforts have often been required to gather a sufficient number of such individuals (indeed, with several hundred students to deal with, the number of such examiners must be high in any event). Without a doubt, however, simply being included in the prestigious club of people involved with the work of mathematics circles becomes an incentive in itself. Consequently, a considerable number of students from universities or pedagogical institutes strive to become involved in such work, even if their own experience with mathematics circles and Olympiads is relatively limited. Mathematics circles are usually run by instructors together with assistants. Starting out as assistants, even individuals who are initially less experienced get a chance to acquire experience gradually, and sometimes they become instructors themselves, subsequently remaining involved in such work for many years.

The distinctive features of the organization and staffing of the selective forms of working with students, which we have just described, often shed important light on the advantages and disadvantages of the system that has taken shape. We should add that although in recent decades special grants to support extracurricular work have started to appear, such work remains in many respects uncompensated and based on the instructors' enthusiasm and desire for prestige or self-fulfillment.

5.1 *On Mathematics Circles*

Mathematics circles are, of course, the most popular form of extracurricular work. In large cities (Moscow, St. Petersburg, Yaroslavl, Krasnodar, Kirov, Chelyabinsk, Irkutsk, Omsk, and others), mathematics circles occur at a citywide or regionwide level. Study in such circles is intended to take place over several years. Such circles are attended by children from many schools who are, as a rule, ready to spend much time not only on solving problems and studying theory, but also on commuting to the locations where their mathematics circles meet, which can consume a considerable amount of time. For example, in 2001, the gold medal at the International Mathematics Olympiad in Washington, D.C., was won by a student who traveled by train every week to attend his mathematics circle in a different city. The trip took three hours in one direction!

In cities other than Moscow and St. Petersburg, work in mathematics circles usually revolves around a very small number (sometimes one or two) of qualified teachers, who over a period of many years engage in the painstaking work of educating gifted children. The graduates of such circles, however, may be seen among the winners of the International Mathematics Olympiads.

In Moscow and St. Petersburg, there are networks of citywide mathematics circles. In St. Petersburg, these include the mathematics circles of the Physics and Mathematics Center of Lyceum 239, the St. Petersburg Palace for Young Creativity, the Mathematics School for Young People, and the Fractal Network of mathematics circles. In Moscow, they include the mathematics circles of the Moscow Center for Continuous Mathematical Education, the mathematics circles of the Lesser Mekhmat (an evening school at the Moscow State University's Mechanics and Mathematics Department), as well as mathematics circles connected with the major schools with an advanced course of study in mathematics. Study in mathematics circles supplements study in schools with advanced courses in mathematics. Usually, students of grades 8–11 who participate in mathematics circles also attend such schools.

The citywide circles bring together hundreds of students (at the time of the writing of this chapter, we estimate the number of students participating in the mathematics circles of the aforementioned networks in St. Petersburg to be around 700). These mathematics circles, of course, are not always identical in their strength and their programs. It should be said that circles in large cities, to some extent, compete with one another. Unfortunately, although the explicit objective of most mathematics circles is not to prepare students for Olympiads, but rather to offer them a comprehensive education in mathematics and to develop their gifts, in practice one sometimes encounters situations that are reminiscent of professional sports — the pursuit of Olympiad honors does exist.

Nonetheless, it would be incorrect, of course, to reduce everything to Olympiads. Topics covered by the mathematics circles and examples of first- and second-year problems may be found in Fomin *et al.* (1996). According to our observations, for the first two or three years with the same group of students, mathematics circles usually adhere, to a greater or lesser degree, to the topics and types of problems presented in this book. Subsequently, both the topics and the format used for working with the students begin to vary in accordance with the instructor's personal preferences (to repeat, a mathematics circle may function from grade 5 to grade 11).

As an example, consider the circles of the Physics and Mathematics Center of Lyceum 239 in St. Petersburg. Their participants are students of ages 10–17 (grades 5–11). Thus, a student may attend the same mathematics circle continuously for up to seven years (although, naturally, some mathematics circles may be formed later). The mathematics circles of the Physics and Mathematics Center meet twice a week, once basic school classes end. These meetings may occur in a variety of different formats; for example, they may be organized as:

- Lectures on theory;
- Individual problem solving;
- Discussions on solutions to problems with teachers;
- Solving problems collectively, in groups;
- Analysis of solutions by the instructor;
- Interviews and exams on theory;
- Seminars;
- Student reports, summaries, and independent projects and research;
- Mathematical competitions.

Mathematics circles (especially the strongest ones) consume much time. The program of a mathematics circle is meant to last for approximately 140–150 hours of "general sessions" per year. However, to this must be added no fewer than 80–90 hours of so-called "Olympiad preparation sessions." A single session of a mathematics circle can often last for four hours.

Each session begins with thoroughly hearing out each child's solutions to all the problems assigned to him or her at the end of the previous session. Such work requires the participation of a large number of volunteers — usually older schoolchildren or university students who serve as assistants to the teacher of the mathematics circle. After this, the teacher presents the solutions to the problems on the blackboard, with requisite theoretical commentary.

The sessions are devoted to solving problems in number theory, graph theory, combinatorial problems and problems about games, geometric problems, problems involving inequalities, and so on. From a certain point on, instructors begin inserting sections on theory that resemble (at least in terms of their content) what is ordinarily studied in universities. Students acquire a thorough grounding in geometric transformations (including inversions, affine and projective transformations), discrete mathematics, groups, rings, fields, calculus, elementary general topology and functional analysis, and combinatorial geometry.

As an example, consider the content of the sections on "Elementary Topology" and "Elementary Functional Analysis":

> The topology of the real number line. Topological definitions of the limit and continuity on the real number line. Compactness on the real number line. The general definition of a topological space. Separability axioms, connectedness axioms. Compactness. Topological definitions of the limit and continuity. Homeomorphisms. Metric spaces.
>
> Complete metric spaces. Quotient spaces of topological and metric spaces. Normed spaces. Banach spaces. Closed graph theorem. Open mapping theorem. Hahn–Banach theorem. Geometric and analytic application of topological ideas and methods.

We should stress, again, that mathematics Olympiads and other competitions lie outside the scope of this chapter. However, they occupy a very prominent place in the activities of mathematics circles, not only as points of reference, sources of problems, and measurements of achievements, but also as a continuous form of work. Olympiads among mathematics circles are a regular occurrence, as are "math battles" within a single mathematics circle and among different circles, and so on. All of this undoubtedly contributes to the formation of future winners of national and international Olympiads.

It must also be stressed, again, that mathematics circles vary. Not all mathematics circles, even if they are attended by students drawn from a whole city, achieve the highest results. On the other hand, it would not be mistaken to say that the system of mathematics circles, say, within the city of St. Petersburg every year produces about 10 (sometimes more, sometimes less) almost fully formed young mathematicians with a mathematical education that is very good for their age. To this must be added the annual inflow of literally hundreds of students from mathematics circles into specialized mathematics schools, the core of whose student bodies is largely composed of these students.

5.2 *Mathematics Summer Camps*

Another component within the structure of multiyear mathematics circles is intensive summer classes, which take place in so-called mathematics summer camps. In the USSR, there were camps for Young Pioneers in the countryside where students could go on their summer vacations. These camps were cheap, since they were supported by the state (usually through various organizations). During a single camp session, which usually lasted three weeks or slightly longer, students would be fed, given a place to sleep, and offered an array of recreational and health-improving activities. At a certain point, there developed a tradition of organizing a mathematics summer camp on the grounds of some camp of the sort just described. During the post-Soviet period, many Young Pioneer camps were destroyed, while the ones that survived were reorganized. However, the tradition of mathematics summer camps survived.

Ordinarily, teachers of the mathematics circles, after arranging a place and time for a mathematics summer camp, assiduously invite the participants of their mathematics circles to attend the camp. Life in the camp is sufficiently close to, say, life in a boy scout camp, except that the mathematics cohort (if the camp is not entirely mathematics-based) does mathematics while the rest of the children go on a field trip, participate in nonmathematical clubs, or simply play or run around the camp. The mathematics cohort might spend, say, six hours daily doing mathematics.

It is probably unnecessary to discuss the program of these summer classes — by and large, they are structured in the same way as ordinary mathematics circles, which is to say that the students solve and analyze a great number of problems. Sometimes during summer classes, teachers make presentations of a theoretical nature on different topics in mathematics. It is evident, in any case, that these three weeks of intensive study are of great importance for the mathematical growth of the students who attend these camps.

5.3 *Conferences*

A form of extracurricular work that is in some sense the opposite of mathematics Olympiads is student conferences. While in the Olympiads the athletic-competitive element is emphasized, the aim of student conferences is to encourage the students' scientific work and bring it to conclusion. Reports, which at one time were the main form of work in mathematics circles, now reappear but in a different capacity. Ideally, the students report about their own results.

We will not attempt to present a reliable and comprehensive history of student conferences in mathematics here, but we can note the important role played by the so-called Festivals of Young Mathematicians, which were held for many years in Batumi, thanks to the energy and initiative of a local teacher, Medea Zhgenti, with the support of the editorial board of the magazine *Kvant*. During the 1970s and the 1980s, these events were held in November, during school vacations, and were attended by teams of students from different cities of the Soviet Union. The program included many reports by students which were heard by the participants and a jury, whose core was usually composed of members of the *Kvant* editorial board.

With the collapse of the Soviet Union, the festivals in Batumi ended, but other all-Russian or municipal conferences appeared (for example, in St. Petersburg, conferences were held around a number of schools, such as Physical Technical School #566 or the Anichkov Lyceum). At a certain point, this format became extremely popular.

The preparation of a report requires systematic and orderly work not only by students but also by their teachers and advisors. The stages of the preparatory process include the preparation of presentations for a class or a mathematics circle based on existing publications (effectively, the retelling of these publications), the assembly of a bibliography on some topic, the preparation of a summary paper consisting of a compilation of several different publications, and so on. A school — at least a school with an advanced course in mathematics — must work to develop in its students the corresponding skills (Karp, 1992). Still, the preparation of a report for a prestigious conference usually requires more than this: namely, an independent result, however minor. This presupposes individual work with a scientific advisor, who poses a problem and guides the student.

As already noted, the tradition of research mathematicians working with students in Russia is very strong, and it has usually been possible to provide for such guidance, at least in schools with an advanced course in mathematics and for the strongest students. At a certain stage, a paradoxical situation arose — although one that did not last long — in which, following a sharp drop in the economic position of university employees, certain secondary schools could to a certain degree finance their work with schoolchildren. Against the background of standard rhetoric about the need to modernize, involve schoolchildren in science, and so on, the number of schoolchildren involved in writing papers of some kind with the support of their scientific advisors increased.

It would probably be impossible to characterize these developments as purely positive or purely negative. If we cannot doubt the usefulness of students doing independent work (even if they do receive some strategic suggestions from their advisors) and generalizing various theorems from the school curriculum or transposing them or similar results onto some other objects, then the expediency of the early study of various typical college-level topics may sometimes be open to question. This touches the well-known issue of the opposition between acceleration and enrichment, about which the least that can be said is that acceleration must be motivated and not without limits. Still, at student conferences, one could hear successful and interesting reports on functional analysis, group theory, and topology.

As a whole, of course, the popularity of conferences, even at their peak, was always noticeably lower than that of mathematics Olympiads. There have been some attempts to combine these two forms of events. At the conferences of the Tournament of the Towns, currently one of the most important international competitions, participants are given so-called research problems, i.e. problems that they have to solve over a comparatively long period such as a week (examples of such problems appear in Berlov *et al.*, 1998).

This experiment, in our view, is of great importance. In general, all of the qualifications formulated above notwithstanding, the role of conferences in extracurricular work with students seems very significant.

6 Conclusion

Naturally, it has been impossible in the space of this chapter either to provide a systematic history of extracurricular work in Russia or to describe all of its forms, let alone to name all those who played a significant role in its development. We have only briefly described the system that has existed and exists at present. However, it is important to emphasize once more that the system's strength lies in its traditions, which have spanned many years, and which appeared and developed largely because the system as a whole was open to anyone interested (even if in recent years critical remarks have often been made that high-level Olympiad participants form a closed club).

The teaching of the mathematically gifted is often contrasted with mass education, although the two are in fact firmly connected (Karp, 2009). It is somewhat naive to repeat, as was once commonly done in Russian periodicals, that "all children are talented": it is unlikely that all children are equally talented in mathematics. However, to identify those who truly have mathematical talent, real mathematics must be offered to all students. Consequently, if in some country few students want to study mathematics in graduate or even undergraduate schools, then this is not so much a consequence of the fact that not enough work has been done with the gifted, as it is a consequence of the fact that not enough work has been done with all students, and that the work done was not done right.

From this point of view, the bottom levels of extracurricular work in mathematics become essential. Naturally, the problems given to students in such cases are not as beautiful and substantive as the ones offered at the top levels of the educational system. Nonetheless, it is precisely broad-based extracurricular work that has made it possible to continue attracting new people to mathematics, not least by creating and supporting a positive public attitude toward studying mathematics.

The current situation in Russia is not all rosy, and difficulties stem not only from economic issues or from the very high rate of emigration among mathematicians, which is destroying — or at least weakening — the traditional ties between school education and the scientific world. There are also internal problems, which include excessive competitiveness. When students concentrate on studying in mathematics circles from an early age and for many years afterward, they may potentially find themselves cut off from the world. This may create problems for graduates of mathematics circles, not only in their social lives, but also in achieving mathematical results, by narrowing their horizons. These and other problems have often been discussed and continue to be discussed within the mathematics community.

At the same time, the achievements of the Russian system of extracurricular work are obviously great. If not absolutely all prominent Russian mathematicians, then certainly an overwhelming majority of them, have passed through this system; wonderful collections of material for schoolchildren have been created within this system; and, most importantly in our view, this is a system that has helped millions of students to become better acquainted with mathematics and to fall in love with it.

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10

On Mathematics Education Research in Russia

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1 Introduction

Some decades ago, a number of important Russian scholarly works on mathematics education were translated into English and thus introduced into the international scholarly discussion (see the first volume of this monograph: Kilpatrick, 2010). This chapter in a certain sense continues what was done then, although its orientation will be somewhat different. The discussion below will likewise address Russian research in mathematics education, although readers will be presented, naturally, not with complete translations of the relevant works, but only with very brief summaries of them. But if the purpose of *Soviet Studies in the Psychology of Learning and Teaching Mathematics* was to acquaint the English-reading audience with the summits of Russian research, and if the collection included translations of works by Davydov, Krutetskii, Menchinskaya, Yakimanskaya, and others, then our goal, in keeping with the general title of this volume, is to demonstrate the current state of Russian research in mathematics education. What characterizes current research — indeed, what perhaps characterizes scientific research in many countries today — is its great variety: it is impossible to point to a single, unified level of work. Consequently, the present chapter refers to and comments on works of widely different levels, addressing both research that in our view is very significant and, conversely, research about which we have doubts. The purpose of the present chapter is not to judge and criticize or praise, but merely to describe the basic issues that are studied and to acquaint readers with the methods employed to study them.

Therefore, we will first address certain general features of the organization of scientific research in mathematics education in Russia and certain general features of our sources, describing the principles that governed the selection of the scientific works that will be briefly discussed below. Then we will turn directly to these works, grouping them according to their orientation and topics. Note that we will focus almost exclusively on a relatively recent period, beginning in 1990, but even given this narrow focus, we make no claim to achieve exhaustiveness — some works, possibly including very important ones, have inevitably been left out of the discussion.

2 On Certain Features of the Organization of Scientific Research in the Area of Mathematics Education and on Our Sources

Traditionally, scientific research on pedagogy in Russia has predominantly taken place in scientific research institutes as well as universities and pedagogical institutes (universities). Several institutes of the Russian Academy of Education have subdivisions that study questions connected with mathematics education. Many universities, pedagogical institutes (universities), and scientific research institutes have graduate schools which prepare scientific researchers.

The system of supporting research through grants disbursed by various foundations was practically nonexistent in the USSR, where research was done either because it was directly commissioned by the government or because it was part of the planned work of an agency, which in turn was supported by the government; or it was done simply at the researcher's own discretion and, one might even say, in the researcher's own free time (to be sure, the social status of, say, a university professor was enhanced when he or she published serious new articles). In mathematics education, this old system has largely endured to this day. Comparatively large scientific and practical research teams usually form around various new curricula and most of their efforts are aimed specifically at developing and testing new textbooks.

Periodicals in which scientific studies of mathematics education can be published are few in number. In Russia, only one traditional (not purely electronic) journal specializes in this field: *Matematika v shkole* (*Mathematics in the School*). It is probably fair to say that the most natural place for many researchers to publish their work today is in regularly published volumes of collected papers and conference proceedings, such as Orlov (2008) or Testov (2007). Separate books devoted to various studies are also published. Certain publications of this type will be discussed below, but for the most part our attention will be focused on dissertation research, which in our view provides a good opportunity to obtain a picture of the topics and nature of Russian research in mathematics education.

We should note at once that in Russia (USSR), in contrast with the United States, for example, in addition to the "Candidate of Science" degree, which is equivalent to the Ph.D., there is a higher degree, the "Doctor of Science" (often translated as "*Dr. Habilitatis*"). A Russian doctoral dissertation, according to a government resolution adopted in 2002 concerning the rules for awarding academic degrees, must contain "either a major new scientific advance or solve a major scientific problem"[1] (http://vak.ed.gov.ru). Not surprisingly, doctoral dissertations are usually defended by mature researchers with a sufficiently large number of publications to their name. Generally speaking, for a person to become a full professor, it is highly desirable that he or she have a Doctor's degree (although one can cite examples of Doctors who are not full professors and vice versa).

Publications are usually a requirement for not merely a Doctor's degree, but also a Candidate's degree. In general, one may say that

[1]This and subsequent translations from Russian are by the authors.

certain requirements which, for example in the United States, must be met by persons seeking to obtain a tenured position — a body of publications, favorable references — are in Russia presented before those who aspire to obtain a Candidate's degree.

The awarding of academic degrees is considerably more centralized than, say again, in the United States. Persons wishing to obtain an academic degree must submit their applications to an academic council (at a scientific research institute or university). After a series of formal procedures, in the event of a favorable outcome a defense takes place at a meeting of the academic council. This is not the end of the matter, however: the material of the case is then sent for review to the so-called Higher Attestation Commission, only after whose confirmation the degree is finally awarded. In addition, both the creation of the academic council itself and the basic requirements that the defense must meet are within the purview of the Commission (http://vak.ed.gov.ru).

The sciences are divided into different areas, and attached to each of them is a six-digit code, whose first two digits indicate the branch of science as a whole. For example, 01.01.01 refers to mathematical analysis, while 01.01.04 refers to geometry and topology. The "methodology of mathematics instruction," which is in certain respects the equivalent of what in English is known as "mathematics education," belongs to the scientific branch of pedagogy (code 13) and to the category 13.00.02 (theory and methodology of teaching and education), which includes the methodologies of teaching other subjects as well. Consequently, an academic council must also have a specialized slant; for example, an academic council might have the right to direct dissertations in category 13.00.02 (mathematics, computer science, physics), but not in category 13.00.02 (Russian language) or category 13.00.01 (general pedagogy and history of pedagogy).

We should say that a large amount of work that is relevant to mathematics education is conducted within the framework of psychological research (the studies that were translated into English in the past were classified in the USSR under the category of psychology, code 19). Such studies, however, fall outside the bounds of our discussion in this chapter.

The current system of dissertation research took shape gradually, but the crucial step — the actual appearance of dissertations in pedagogy in the USSR — occurred in 1934 (Zaguzov, 1999a). Doctoral dissertations in the methodology of mathematics education, however, did not appear immediately: the first such dissertation, *Theoretic Arithmetic* by I. V. Arnold, was defended only in 1941.

Zaguzov (1999b) produced an index that contains the titles of all doctoral dissertations defended in pedagogy from 1937 until 1998 and the names of their authors. The figure below, which indicates the number of doctoral dissertations defended in each decade, is based on the information found in this index. (The index contains some errors — some of the dissertations mentioned there were in fact never defended, and conversely, in some instances, dissertations that were defended are not mentioned in the index. Nonetheless, the number of such cases is very small.) From 1999 until the present time, as far as may be judged from the catalogs of the main libraries, no fewer than 50 doctoral dissertations in the category 13.00.02 (mathematics) have been defended.

As already noted, it is customary to publish the results of dissertation research prior to the defense. In addition to the formats named above, the Higher Attestation Commission recognizes publications in "general" journals, such as the bulletins published by relatively major universities. Persons wishing to obtain an academic degree

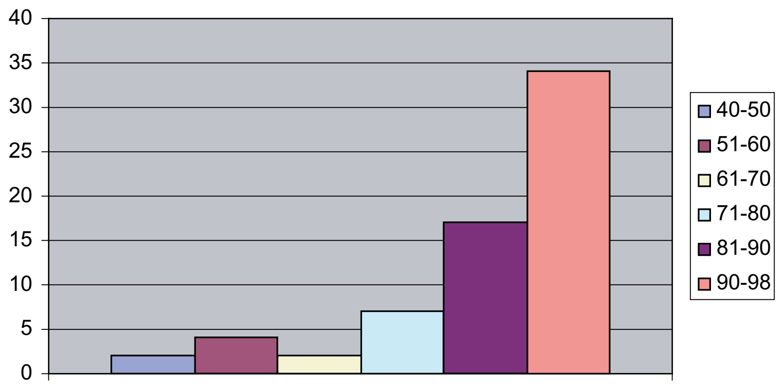

Fig. 1. Defended doctoral dissertations in mathematics education by decade.

must prepare a so-called author's summary of 40,000 characters for a Candidate's degree and 80,000 characters for a Doctor's, containing a brief description of the work that they have done. This text is not considered an official publication; nevertheless, 100 copies of it are published and sent to all of the major scientific centers and libraries in the country.

The authors' summaries of doctoral dissertations in category 13.00.02 (mathematics) constitute the basic material that will be analyzed in this chapter. We have at our disposal 92 authors' summaries of doctoral dissertations defended since 1990, which we have located in the St. Petersburg libraries that are accessible to us (first and foremost, the National Library of Russia). Evidently, not all authors' summaries have been submitted to libraries in St. Petersburg, and not all have been preserved there. Nonetheless, we clearly have most authors' summaries at our disposal (note that databases such as ProQuest Digital Dissertations do not exist in Russia). In addition, we will discuss certain scholarly books on mathematics education, including collections of articles. About Russian Candidates' dissertations, we will say only a few words to convey an idea of what such studies require, what directions they take, and what specific characteristics they share. Most of this chapter, therefore, will be a kind of catalog of the works with brief descriptions of their content. Again, certain omissions and gaps are inevitable. At the same time, we hope that this format will allow readers to identify the works that interest them while giving us material for a concluding general analysis.

In discussing Russian research in mathematics education and dissertation studies in this area, one cannot avoid asking the following question: What exactly should be considered "Russian"? The republics that once formed the USSR have become independent; nonetheless, the "separation" of the various systems for awarding academic degrees was by no means immediate and indeed is still not complete at present. Moreover, researchers from other countries in the Eastern Bloc (for example, Poland) frequently came to the Soviet Union, and have subsequently continued to come to Russia, in order to defend doctoral dissertations. Such studies may or may not be considered as belonging to Russian science — a case can be made for both views. In this chapter, all dissertations defended in Russian before dissertation councils operating on the territory of the former USSR in accordance with the standards of the Russian Higher Attestation Commission (and submitting authors' summaries to Russian libraries) are considered as Russian studies. We deliberately specify, however, that a dissertation is based on material from another country or has been defended outside Russia when this is the case.

Another circumstance that must be taken into account is that along with dissertation defenses based on the submitted text of an extended dissertation, there exist what are known as "report-based defenses." This much more rarely employed format is used when the degree-seeker is the author of many published works, which themselves are the texts to be defended. The degree-seeker may obtain the right, in place of submitting a separate dissertation, to prepare only a comparatively short "report," which includes the usual parts of a dissertation, such as a "need of the study" section. This format is fairly rare, and we will generally not specify the particular manner in which a dissertation was defended.

Below, we arrange texts according to several basic themes. Naturally, any such division and grouping of diverse texts into separate categories must be somewhat artificial. The same text may be included in several different categories. Nonetheless, in our view, such a division helps to reveal the multiformity of topics in Russian mathematics education studies.

3 Issues in the Philosophy and Worldview of Mathematics Education

Soviet mathematics education, like all other scientific disciplines, needed to bow, to some extent, to the reigning philosophical phraseology. Later, references to Marxist–Leninist classics became unnecessary and even to a certain degree contrary to the accepted style. Nonetheless, interest in discussing general philosophical questions remained. In this section, we will mention two related works.

The first of these (Ivanova, 1998) is devoted to the problem of the so-called "humanitarization" of education, by which is meant "reviving education's culture-creating function, giving education a 'human dimension,' endowing the students with a holistic understanding of man and society" (p. 4). Humanitarization was envisioned as a contrast to the technocratic approach, which was criticized for conceiving of human beings as elements in a machine, carrying out prescribed functions.

This terminology, however, apparently failed to become universally understood; at least, Ivanova (1998) notes that 69% of mathematics teachers were unable to explain what the humanities-type aspects of mathematics education consisted of. In fact, the aim of Ivanova's work was precisely to develop a conception of the humanitarization of mathematics education. Consequently, her work is theoretical in nature and connects the problem of humanitarization with the problem of personality development. As she writes, "humanities-type knowledge is knowledge that has been acquired by the students themselves in the course of intensive intellectual–emotional exploratory activity" (p. 20). She declares the structure of the personality and the patterns of its development to be the key elements in her model of humanitarization (p. 24). On a somewhat more practical level, Ivanova recommends involving students in creative mathematical activity, underscoring the aesthetic side of mathematics, and using historical material. She similarly characterizes mathematics education methodology from the viewpoint of the humanitarization of education. To describe the aims of such education, she employs Bloom's (1956) taxonomy.

The work of Zhokhov (1999) raises even more general questions: the topic here is the formation of a worldview in mathematics classes. He builds his conception axiomatically, formulating eight postulates. Without repeating them here, let us note that his work emphasizes that the central purpose of mathematics education is defined by (1) the methods and means of learning that are specific to mathematics and (2) the view of the world that is specific to mathematics (p. 26). Consequently, he proposes that attention should be focused not so much on the content of the school curriculum (which has inevitably fallen far behind the development of the discipline) as on the methods of mathematical activity that are demonstrated when mathematics is studied.

4 The Psychology of Mathematics Education

Russian (Soviet) psychologists have devoted much attention to problems connected with mathematics education. Of the studies that have appeared in recent years, the first that must be mentioned is Yakimanskaya's (2004) manual, which sums up many years of research by the author and her students — above all, research pertaining to spatial reasoning. No detailed analysis of this and other psychological studies can be undertaken here — we have already noted that all of the texts discussed in this chapter are "officially" considered pedagogical, not psychological. In this section, we will discuss relatively few works, although psychological studies are used and cited in virtually all studies in mathematics education as well. Nonetheless, we have set apart this section to discuss works whose central aim is to study psychological characteristics.

The psychological foundation of practically all contemporary studies (at least, according to what their authors themselves state) is Vygotsky's conception of the developmental function of education. Stefanova *et al.* (2009) point out that the "contemporary education system is oriented to a greater extent around the developmental aspect of education than around its informational aspect" (p. 67). However, the question of what developmental education in mathematics comprises, both in general theoretic and in practical terms, continues to be discussed from various angles.

Ganeev (1997) defines it as follows: "... education whose purpose and outcome lie in the formation of new mental structures in the students, which allow them fully to assimilate knowledge" (p. 15). The version of developmental education which he describes is a system based on what he calls the "informational–developmental method"; this system includes a whole range of measures, including measures aimed at "increasing the informational–cognitive load of the problem-solving process" (p. 13) and so on. Consequently, he identifies a set of conditions under which education can be successful. His basic assumption is that students must take part in the process of posing cognitive problems and reflecting on cognitive-learning activities. His theoretical constructions are supplemented with programmatic–methodological recommendations, whose practical value is buttressed by experimental data. These data, as Ganeev writes, have demonstrated a noticeable improvement in students' performance in experimental classes in ordinary school mathematics subjects (particularly geometry, which, as he explains it, is a subject less grounded in algorithms and more creative than algebra), as well as in the solving of problems on tests aimed at determining the level of students' intellectual development.

The work of Reznik (1997) also pertains to a certain extent to research on developmental education, but what is investigated here is a specific aspect of it: the role and development of visual thinking (as she calls it). Following the well-known Russian psychologist Zinchenko, Reznik defines visual thinking as follows: "… an activity whose product consists in the emergence of new images, the formation of new visual forms, which carry a certain conceptual weight and render meaning visible" (p. 10). Another important concept for her is visual translation, i.e. the deciphering of incoming data through the process of visual perception with the help of a reserve of familiar forms or terminological denominations. Further, she discusses how a visual educational environment (i.e. conditions in which visual thinking is actively employed) can be organized and put to use in mathematics education. In this context, she proposes special formats for working with visual materials (informational schemas, informational notebooks). She also discusses methodological questions, including questions concerning the visual search for the solution to mathematical problems (i.e. questions concerning the process of emerging new visual forms). The concluding chapter of the study is devoted to a description of experimental work carried out in accordance with the researcher's theoretical position.

Tsukar' (1999) investigates a related topic — thinking with images ["thinking whose main function is operating with images" (p. 10), as the researcher explains]. After demonstrating the importance of such thinking in theory, the author presents a large number of techniques and methods for developing such thinking (he even describes a special device for constructing problems). In conclusion, as in the Reznik study described above, he presents data on pedagogical experimental work.

Pardala (1993), in a study written even earlier and based on Polish material, investigates the problem of "mathematical seeing" (p. 6) and, spatial imagination, in particular. Discussing the importance of developing these features, he relies on a large body of psychological and methodological studies, which have demonstrated on the one hand the importance of developing an informal, intuitive understanding of geometry, and on the other hand the fundamental physiological origins of the difference between image-oriented and algorithmic–logical thinking. He notes that when the formation and development of the spatial imagination is ineffective, this is due mainly to an imbalance between theory and practice in the teaching of geometry (specifically, insufficient attention to problem solving). On the basis of this and a number of other approaches, he analyzes the manner in which the spatial imagination develops in actual practice in basic school; he also elaborates a conception of how the spatial imagination develops within a framework of differentiated mathematics education. Considering such development as a unified and continuous theme of the school course in mathematics, Pardala formulates a variety of methodological recommendations, including a typology of, and a set of general principles for, problems aimed at facilitating such development.

The work of Lipatnikova (2005) is also concerned with the problems of developmental education. This author highlights the role of the *reflexive approach*, in which "students investigate, interpret, and reinterpret information, transforming it by independently choosing microgoals" (p. 16). More concretely, she studies the application of the reflexive approach to the use of oral exercises. She identifies the various functions that such exercises have in the learning process and proposes a model of the reflexive approach that employs such exercises (to use her own terminology). This model includes such stages as solving exercises using an already-known technique, criticizing a technique used earlier, and constructing a new technique. Lipatnikova is the author of numerous collections of oral exercises for grades 1–6.

Malikov (2005), whose work is based on material from Kazakhstan, sets for himself the ambitious goal of "developing a theoretical model of and practical recommendations for defining the relation between intuition and logic in mathematics education, with a view to facilitating an increase in the effectiveness of education" (p. 5). The author's theoretical investigation as well as his practical observations led him to conclude that the role of the intuitive must be augmented, while increasing logical rigor negatively affects students' involvement in learning. For example, he cites the results of an analysis of actual school practices, which indicate that even with imprecise mathematical definitions students form accurate conceptions thanks to their intuition. At the same time, he recommends increasing the quantity of learning material not "by omitting 'intermediary stages,' but by accelerating its presentation" (p. 31), particularly by making use of historical material.

The goal of Egorchenko's (2003) study is "to develop a conception of how students form and develop notions of the essence of mathematical abstractions" (p. 8). The researcher characterizes the body of problem situations and material that facilitate the formation of such notions as "methodological reality" and describes it by using such concepts as teaching goals, interconnections with teaching practice, and modeling. Consequently, Egorchenko devotes considerable attention to the applied aspects of mathematics education and to modeling.

5 Problem Solving

The basis on which the studies in this section are grouped together and isolated from the rest is also somewhat artificial: problem solving may be considered one of the principal themes of all of Russian research. In one way or another, it is mentioned in virtually every paper on mathematics education. Prior to the period discussed here, many books appeared that were wholly devoted to problems and the theory of solving them (such as Friedman, 1977; Kolyagin, 1977; Metel'sky, 1975; Stolyar, 1974). Problems have been studied from the most varied angles: several systems have been proposed for classifying problems; a notion of problem "complexity" (as an aspect of the problem itself) has been defined; the "difficulty" of a problem has been quantified as a psychological–pedagogical characteristic (for example, as inversely proportional to the number of students who have solved the problem); and the psychological, informational, and structural components of problem solving have been identified (Krupich, 1992). These and other aspects of research concerned with the phenomenon of school problems and their history have been discussed in Zaikin and Ariutkina (2007) and Shagilova (2007).

Some time ago, Sarantsev (1995) studied a concept that, in his terminology, was narrower than a problem: the exercise ("a problem is an exercise if it results directly in the acquisition of new knowledge, skills, and abilities" (p. 17), according to his definition). He regards exercises as the effective vehicles of learning and proposes structuring the whole education system on exercises. Exercises, according to him, constitute a means of efficacious and goal-directed student development (pp. 11–13).

The structure of exercise sets has been studied by Grudenov (1990). In particular, he focuses on the contradictions inherent in using exercises of the same type: stable skills cannot be formed without them, yet their use leads to diminished interest. He sees the solution in the combined use of a variety of different teaching principles.

The findings of recent studies in the area of problem solving are described in the proceedings of a special conference devoted to problems (Testov, 2007).

As for dissertation research, Krupich (1992) aims at "developing a theoretical basis for teaching school-level mathematics problem solving" (p. 5). The key words for this study are probably "systemic," "cohesive," and "structural." Krupich views the problem as a complex structure or, more precisely, as a conjunction of two structures: an external structure, i.e. the problem's actual conditions and the information given; and an internal structure, which includes the problem's substantive characteristics (including its difficulty). The structural unit of the learning process, according to him, is the "instructional problem with a three-part structure: the problem itself, the students' cognitive contribution, and the didactic technique used by the teacher" (p. 15). Krupich analyzes existing textbooks and finds that the problem sets in them are incomplete, not hierarchically structured in terms of their difficulty, and so on. (He precisely defines and elaborates on all of these concepts in his study.) Furthermore, he also proposes his own classification of problem-solving techniques.

Ryzhik (1993) also addresses what the system of problems contained in a school textbook should look like. His conception includes the principle that the system of problems must be interconnected with (1) the environment (for example, it must take into account social needs, the state of the various sciences, etc.); (2) the theoretical material in the textbook; (3) the teacher (for example, by allowing individual teachers to select what they need); and (4) the student (for example, by providing for the development of each student). In fleshing out these principles, Ryzhik proposes several requirements or objectives for the system of problems contained in the school problem book on geometry, beginning with the objective of having the problem book reflect contemporary views of geometry, and continuing with the objective of forming foundations for research-oriented activity and invention, as well as the objective of giving students material that corresponds to their development at any given point and material that can facilitate their further development. In formulating his theoretical position, he relies on his experience as the author of numerous textbooks.

Voron'ko's (2005) aim is to research students' investigative activity in the process of mathematics education, to which end she studies students' problem-solving activity. Identifying what she considers to be the basic types of investigative activity developed in the process of mathematics education (such as posing problems and formulating hypotheses), she demonstrates how they may be developed using problems. Consequently, considerable attention is devoted to classifying problems and to discussing specific types of problems.

6 The History of Mathematics Education

Mathematics education in the USSR could not, of course, remain wholly unaffected by the ideological campaigns that occurred in the country. Nonetheless, because the government recognized the importance of the subject for the country's industrial and military development, the teaching of mathematics likely suffered less than other areas from ideological pressure (Karp, 2007). The history of mathematics education, however, belonged to a different category — history — in which "the unprincipled and the unideological" or "objectivism" was generally not supposed to exist. It would, of course, be overly simplistic to conclude that "objectivism" really did not exist. We can point to a number of sound and serious works that appeared during Soviet years, which were based on the thorough study of archival material (for example, Prudnikov, 1956). However, many works are of an entirely different nature, and today's researchers can thus approach this material from the vantage point of different traditions.

We will begin this section with the studies of Polyakova (1997, 2002). They can be judged to some degree by the chapter she wrote for the first volume of this book, which is based on the works just cited. She has written what is probably the only systematic course in the history of Russian mathematics education — from the birth of the Russian state until the Revolution of 1917 — that is accessible to the general reader today. Her works take into account the conclusions and findings of several generations of historians of mathematics and mathematics education, and also include examinations of numerous educational manuals.

Polyakova's doctoral dissertation (1998) is devoted not so much to the history as to the historical preparation of mathematics teachers. The aim of her research is "to provide a theoretical and practical foundation for the need to make … historical–methodological preparation a part of the professional preparation of the mathematics teachers, and also to identify the conditions that make such preparation effective" (p. 9). Consequently, relying on numerous works on teacher education, she demonstrates the usefulness of a special course in the history of mathematics education. She also proposes several characteristics that such a course should have, including her own periodization of the development of mathematics education. Polyakova concludes by citing an experiment involving interviews with numerous respondents to demonstrate significant improvement in students' historical–methodological competence as a result of taking a course in history.

The work of Yuri Kolyagin (2001), a member of the Russian Academy of Education, is structured as a lecture course in the history of mathematics education in Russian schools. In contrast with Polyakova's books, Kolyagin gives a prominent place to the history after 1917 and particularly to the recent past, which he witnessed and in which he participated. Consequently, questions concerning education policy are at the center of his attention [it is noteworthy that, as indicated in Kolyagin (2001), the then head of the State Duma (Parliament) Committee on Education, Ivan Melnikov, was a reviewer of Kolyagin's book]. The author's idea may be briefly characterized in the following way. Before the Revolution, schools went through a successful evolution and, by 1917, they had reached a very high level of development [as evidence for which the author reproduces the diploma of his aunt, pointing out that "this document vividly illustrates the level of preparation in secondary educational institutions" (p. 134)]. However, unfortunately, "left-leaning parties, mainly socialists, got the upper hand. As is also well known, the leadership of these parties was predominantly non-Russian" (p. 139). "Homegrown Masons, who were virtually agents of Western influence," along with the "products of the provincial intelligentsia" who had filled up the cultural vacuum (here, a reference to Lenin makes it clear that the author means the Jewish intelligentsia), strove to destroy the existing order along with the whole great spiritual legacy of the Russian people (p. 139).

Consequently, Kolyagin's characterization of schools after the Revolution is unequivocally negative. For the radical restructuring of schools during the 1930s and the return to pre-Revolution models, he expresses "thanks to the Soviet government" (p. 161). The following 20 years are described as a golden age of stability, and new reforms are subsequently labeled as a "storm" (p. 172) and "expansionism" (of Bourbaki and Piaget, see p. 191 and p. 194, respectively); while the events of recent decades are characterized simply as "spiritual aggression" (p. 236). In conclusion, the author again turns to general issues, explaining that a great divide "runs along the line between East and West." On one side of this line stand Russian nationalists and patriots, on the other are Westernizers — those who "accept no ideals (except the 'golden calf')" (p. 251).

The work of Avdeeva (2005) is structured around historical material, but her main aim once again is "to develop a methodology for the preparation of mathematics teachers ... based on the lives and works of great educators" (p. 4). The great educators chosen by Avdeeva are K. D. Kraevich, the author of numerous pre-Revolution physics textbooks, and A. P. Kiselev, the author of the most popular mathematics textbooks both from before the Revolution and during the Soviet period. Both of these individuals were originally from the Orlov region, where Avdeeva herself works. Consequently, her dissertation on the one hand describes how the study of the personality of a famous educator may be structured, both in school and in a pedagogical institute or university (for example, she provides lesson plans); on the other hand, it offers a description of the lives and careers of Kiselev and Krayevich. By studying archival documents, Avdeeva has been able to establish many of the details of Kiselev's childhood, such as the names of his own teachers. She has also succeeded in finding certain methodological articles by these teachers, which in her view had an influence on Kiselev.

Kondratieva (2006) sets herself the goal of formulating a "comprehensive conception of the development of mathematics education in Russian schools during the second half of the 19th century" (p. 6). Her work discusses a great deal of factual material, including archival data and articles from periodicals published during the period under investigation. Pointing out that this period witnessed a significant expansion of the education system, as well as an improvement in the methodology of the teaching of the mathematical sciences — not to mention the creation of such a methodology — the author inquires into the dominant philosophical aspects of these developments. Her view is that three basic conceptual components may be identified (pp. 15–16): (1) the recognition of the importance of mathematics as a subject independent of the general orientation of education (be it classical — devoting considerable attention to ancient languages — or real school education); (2) the emphasis placed on general character-building in the process of mathematics education (Kondratieva mentions the cultivation of modesty, orderliness, and diligent work habits, as well as the cultivation of religious feeling); and (3) the notion that the modernization of school education must be based first and foremost on Russian research and solutions. In particular, the author mentions the importance of fighting against "German" influence (a different perspective on the discussions that took place at that time is presented in Karp, 2006).

Kondratieva also analyzes the findings of the methodological science of that period, identifying what she sees as its main currents and ideas. In conclusion, she carries out a comparison between the schools of the second half of the 19th century and the schools of today.

Savvina (2003) has carried out a systematic study of the development of the teaching of advanced mathematics (analytic geometry and calculus) in Russian secondary schools. She has analyzed many archival materials, including the reports of educational institutions and their inspectors, class registers, and the dispatches of school district overseers, as well as contemporaneous periodicals, sources on the history of specific institutions, school curricula and syllabi, and textbooks and teaching manuals. The author begins her account with the 18th century and follows it practically to the present day, identifying various periods and stages in the teaching of the elements of advanced mathematics. In the process, Savvina establishes many concrete historical details and analyzes various approaches employed in school textbooks and teaching manuals.

Among recent studies that make use of a large number of diverse primary sources, we should mention the work of Busev (2007, 2009), which examines mathematics education during the 1920s and 1930s. Busev devotes particular attention to the discussion of issues connected with mathematics education in the press and provides a selection of data about what went on in actual classrooms.

In concluding this section, we should mention two studies whose subject matter lies at the intersection of mathematics education and other pedagogical fields. Petrova (2004) has studied the formation of the system of bilingual education in Yakutia on the example of mathematics education. Her work is devoted mainly to bilingual education and to related general questions, but it also contains sections that are of interest to the historian of mathematics education. In particular, she offers a periodization of the development of education in Yakutia and identifies such important periods as 1918–1923 (when teaching Yakutia students in their native language started) and 1963–1965 (when, on the contrary, the teaching of mathematics in Yakutia was halted).

The work of Zharov (2002) draws on his experience in teaching Chinese students at an engineering college in Moscow. He connects his teaching with an analysis of medieval Chinese mathematics literature, to which end he in turn "develops and deploys elements of constructive mathematics (theory of algorithms) in modeling the content of texts" and so on (p. 7). Consequently, among his principal achievements, the author mentions that he was "the first to propose the formalization of scientific–pedagogical texts as a technique" (p. 12), and even claims that "it is in principle possible to describe the processes of student learning and thinking in pedagogical practice using the methods of constructive mathematics" (p. 13). He devotes considerable attention in his work to assembling different kinds of dictionaries and varieties of programming languages, which according to him adequately represent the cognitive processes of the authors of ancient Chinese tractates.

7 Issues of Differentiation in Education

Russia (USSR) has extensive experience in organizing multilevel education. Gorbachev's *perestroika* and the period that followed, which emphasized the value of the individual, revitalized interest in the subject of differentiated education. Gusev (1990) examines the problem in general terms. He identifies three broad aims of mathematics education: to give students a robust education in mathematics, to facilitate the formation of their personal qualities, and to teach them to apply mathematical knowledge effectively and communicate mathematically. Subsequently, he devotes considerable attention to the second of these aims, which includes the development of students' scientific curiosity, mental development, and so on; more broadly, he looks at the methods for differentiated education in mathematics. In particular, he discusses a system of independent projects for students and the selection and construction of "chains" of assignments.

Gutsanovich (2001) elaborates a broad conception of mathematical development (as a part of general mental development) in the context of differentiated education. In this study, completed in Belarus, the author aims to elucidate the very notion of "mathematical development," connecting it with the notions of "mathematical preparation" and "mathematical abilities." Identifying four levels of mathematical preparation (from "insufficient" to "creative"), he juxtaposes them with nine levels of mathematical development — from "infantile," "descriptive," and "formal" to "creative" and "mathematically gifted." He points to a number of factors that can raise the level of mathematical development: organizational–methodological factors, social–psychological factors, psychological–pedagogical factors, and psycho–physiological factors. In addition, he examines the influence of various mathematical assignments on students' development. His work makes use of a large body of experimental material. In particular, he establishes the frequencies with which the aforementioned levels of mathematical development are reached before and after experimental teaching. Also noteworthy is Gutsanovich's conclusion: "The correlation between the grades given in schools to evaluate the level of performance, and the level of mathematical preparation, or the level of mathematical abilities, is absent or weak" (p. 24).

The Polish mathematics educator Klakla (2003) has studied the development of creative mathematical activity in classes with an advanced course of study in mathematics. To this end, he has theoretically researched the concept of creative activity in general and in mathematics, in particular. Klakla identifies the principal types of students' creative activity and discusses the ways in which they form. Specifically, he focuses on the methodology of solving multistage problems in classes with an advanced course of study in mathematics.

The work of Smirnova (1995) also draws on material from specialized classes. She points out that the very notion of differentiation has meant different things at different times and that this term may presently be used with reference to either pedagogical differentiation, psychological differentiation, or methodological differentiation. She herself focuses her attention on so-called "profile differentiation," i.e. differentiation based on the general orientation of subsequent studies (humanities, technology, natural sciences, and so on). Her work deals with classes of different "profiles" that appeared in the late 1980s and 1990s, and the teaching of geometry in these classes. Analyzing various topics of the course in mathematics, Smirnova describes each of them in terms of a vector with six coordinates, which correspond, respectively, to the humanities-oriented content of the topic, to the number of applications that the topic has in other topics, to the number of new concepts associated with the topic, to the number of basic theorems associated with the topic, to the number of supporting problems associated with the topic, and to the number of practical skills that the student must master in the process of studying the topic (the author establishes these values in different ways). In Smirnova's opinion, the values that she obtains are important for determining the role of the given topic in any given "profile" course. According to her, the success of this model of education is confirmed by such important indicators as level of student interest and effectiveness of instruction (determined experimentally). In her conclusion, she addresses the preparation of future geometry teachers.

"Profile" differentiation is also discussed by Prokofiev (2005), who concentrates on classes at technical colleges that were introduced to raise the quality of incoming students. The author partly contrasts such classes with more traditional classes that offer an advanced course of study in mathematics, because in the former "the stress must be shifted in the direction of applied mathematics" (p. 18). He details the content of instruction in such classes and also names several principles on which this instruction must be based (for example, the principle of individual differentiation). In his opinion, experimental data (i.e. data about the work of classes associated with the college where he worked) support his idea, because, for example, the graduates of specialized, precollege classes have much better scores on their college entrance exams than the graduates of ordinary classes.

8 The Organization of the Educational Process

This section addresses studies devoted to the general principles that underlie the writing of textbooks and teaching manuals in mathematics, the organization of the educational process in mathematics under special conditions, the development of mathematics education standards, and the structure of mathematics lessons.

The very word "standard" came to Russian mathematics education relatively recently — in the USSR and other countries of the Soviet bloc, discussion usually revolved around programs that had to be followed very precisely. The word "standard" was, and indeed continues to be, understood in various ways, but in any case, from a normative perspective, it has always been taken to mean something that replaces old notions (this must be borne in mind since, in other countries, standards often usher in some form of additional standardization, whereas in Russia they have replaced a more centralized system). Yaskevich (1992) discusses the theoretical principles that may be used for defining mathematics education standards (for Poland). By "standard" she understands a norm that includes such components as minimum requirements and prospective requirements, as well as minimum content and supplementary material (p. 15). Furthermore, analyzing Polish and foreign studies, she formulates educational aims and describes principles and a mechanism for selecting content. Based on her theoretical work, she has developed a curriculum plan for classes 4–8 in Poland; this plan has undergone an experimental trial, which, according to her, has supported her theoretical propositions.

Zaikin (1993) studies a problem that is important specifically for Russia, with its vast spaces between small population centers: the problem of teaching in very small village schools, i.e. in schools whose classes have very few students (sometimes even only one). Remarking that education under such conditions must be organized in a nonstandard manner, the author offers a formalized description of organizational structure. To this end, he identifies such parameters as methods of grouping (the teacher can work with the whole class, with groups, or with individual students), methods of student collaboration (the author argues that students may work collectively, cooperatively, or individually), and methods of teacher supervision (the students may work under the direct supervision of the teacher, partly independently or wholly independently). This schema allows Zaikin to define the work format at every point in the lesson and to describe the structure of the lesson (as a chain of triplets that characterize each episode). Further, he studies the effectiveness of different formats in classes (on the basis of observations and tests).

Manvelov (1997) studies the structure of mathematics lessons under different conditions. Identifying the most characteristic types of lessons, he divides them into groups. For example, the first group includes lessons devoted to reinforcing what has been learned, lessons devoted to the generalization and systematization of knowledge, lessons devoted to testing and monitoring, and so on. The second group includes lecture lessons, seminar lessons, workshop lessons, and so on; one more group includes competition lessons, simulation exercise lessons, theatrical lessons, and so on. Manvelov notes, however, that quite often, not the whole lesson but only a part of it has a given form. In other words, the lesson consists of several parts, which may be described in the above terms. Further, Manvelov studies the effectiveness of different configurations (relying on teachers' assessments) and looks at the effectiveness of different lesson structures in terms of other parameters (such as the quality of the content chosen for the lesson). His study describes experimental teaching on the basis of the approaches to lesson construction that he proposes, and compares these classes with control classes.

Gelfman's (2004) goal is to construct educational texts that can create propitious conditions for intellectual character-building for students in grades 5–9. Relying on theoretical analysis, she identifies a set of functions that contemporary textbooks must perform (including educational, supervisory, developmental, and other functions). The notion of intellectual character-building is elaborated by the author; for example, she mentions interest in patterns or in searching for unifications as characteristics that are desirable for students to develop). Further, she focuses on the course in mathematics for grades 5–9, attempting to establish theoretically the principal pathways for enriching the students' conceptual, metacognitive, and emotional–evaluative experience in studying such a course. The conclusion of the study is devoted to discussing work on experimental manuals.

Grushevsky (2001) focuses not just on textbooks, but also on so-called educational–informational kits, which include contemporary information and communication technologies. The author claims to have developed a general structure for such kits and the theoretical foundation for their assembly, including suggestions for new mathematics education technologies (p. 12). In his conclusion, he describes the results of teaching in schools and colleges with the use of kits developed according to his methodology.

9 Studying the Process of Teaching Mathematics: Connections Within Subjects, Continuity and Succession in Education

In Soviet methodology, the teaching of mathematics was traditionally seen as being connected with the teaching of other subjects and with establishing and underscoring links between the various topics covered. Dalinger (1992) collected numerous examples to demonstrate that the aim of teaching students to view mathematics as a unified subject has been achieved only to a very small degree. He himself identifies several groups of possibilities for establishing such links, in particular pointing out links offered by the subject itself and possibilities that arise in the course of a teaching activity. As examples of the latter, he mentions a set of various problems and, in general, the involvement of students in a type of activity "that would allow them to assimilate the main components of a concept and its internal conceptual connections" (p. 29). Dalinger's study contains much information on how students solve (or fail to solve) various problems; he analyzes the obtained data and offers general theoretical and concrete methodological recommendations.

To some degree, Sanina (2002) continues in the same line of work, attempting to construct a theory and methodology for generalizing and systematizing students' knowledge. While noting that generalization and systematization may also occur spontaneously, she searches for forms of working with students that might help many (if not all) of them to acquire not fragmentary but systematic knowledge. Her approach to solving this methodological problem consists largely in constructing special lessons devoted to generalization. She works on the methodology (and theory) of such lessons, formulating, for example, the criteria for selecting systems of problems for such lessons or defining the degree to which students' knowledge is systematic and the degree to which students have assimilated generalized knowledge. Sanina also examines the possibilities of constructing special courses devoted to integration. She writes that her experimental work on the methodology of generalizing spanned 13 years and encompassed both diagnostic and formative stages (during which she determined how generalization usually takes place and shaped a new approach), as well as a concluding stage devoted to monitoring and verification.

Links between topics studied at different organizational stages constitute a special class of links within a subject. In this context, it is customary to speak about *continuity* in education. Turkina (2003) studies continuity within a framework of developmental education. She takes developmental education to mean, first and foremost, education in which attention is concentrated on students and not on the educational process. Her analyses of existing data once again demonstrate that continuity is a critical problem: during the transition from elementary school to the first grades of middle schools (to use Western terminology), students' grades noticeably drop, and the same happens during the transition to a different form of subject organization in mathematics education (in seventh grade). Among the theoretical results of her analysis, we should note that she considers it expedient, in addition to distinguishing between a "zone of actual development" and a "zone of proximal development" (in which, according to Vygotsky, education must take place), to identify a "zone of prospective development," in which education will take place in the future. This zone must be assessed and prognosticated in order to establish continuity in education. Turkina formulates concrete recommendations for teachers, including the suggestion to create situations in which students can construct the necessary knowledge and establish the necessary continuity links on their own. She has carried out experimental work which, according to her, has confirmed her propositions.

Magomeddibirova (2004) likewise focuses on issues of continuity, but she concentrates on the development of a concrete methodology for achieving continuity as students acquire computational literacy in studying algebra and geometry, and solving word problems. The overall conception of the approaches which she recommends includes, for example, the suggestion that "each stage of education be oriented around the scope and level of the students' previously acquired knowledge" (p. 16).

Kuznetsova (2006) contrasts between students' knowledge, abilities, and skills acquired in secondary schools and the system of knowledge, abilities, and skills that are indispensable for successful study in college. The aim of her work, therefore, is to achieve some unity in education. Her study relies largely on educational materials from so-called preparatory studies departments which prepare foreign students for entering college. Criticizing existing textbooks for logical and methodological gaps, she presents a number of ideas, such as the importance of integrating different kinds of subject knowledge, the importance of historical and logical unity in education, as well as the importance of combining a broad-view approach with an algorithm-based approach. She proposes a "dynamic model of the educational process in ... the preparatory studies department" (p. 41) and specifically examines a special goal-directed function with such parameters as I_1 — the teacher's interest in the process of teaching; I_2 — the students' interest; and many others (pp. 34–35). The expected pedagogical effects of her program have been tested in an experimental course designed in accordance with her general theoretical propositions.

The issue of reinforcing acquired knowledge, related to the issues examined above, is the focus of Imranov's (1996) dissertation. This study, which draws on material from Azerbaijan, devotes considerable attention to analyzing the existing literature on the subject, as well as, for example, discussing methods to reinforce knowledge such as independent projects. According to the author, he has developed a new methodology for reinforcing knowledge.

The work of Kozlovska (2004), which draws on Polish materials, is aimed at "developing a pedagogical foundation for assessing and prognosticating students' educational achievements in mathematics" (p. 3). Relying on observational data, she argues that ordinary school grades are subjective. As a supplementary technique, she proposes the use of testing methodologies that were new to the countries of the former socialist bloc. Kozlovska discusses in detail the methodology of constructing and applying tests. She provides interesting data about students' results, on the basis of which she argues that there are significant differences between grades received in basic schools and in lyceums (the next educational institution, corresponding to high schools), and also that students' grades remain more stable in higher grades.

The work of Episheva (1999) is devoted to organizing mathematics education in a way that is oriented around helping students to form skills associated with learning activities. She identifies four groups of such skills: general educational skills, general mathematical skills, specialized skills in different mathematical disciplines, and specific skills formed in association with specific topics. General educational skills include memory organization skills, skills connected with independently working with the textbook, speech development skills, and so on. In Episheva's view, along with strictly mathematical content, educational material must include the description of activities that aim to teach this mathematical content (p. 36). The stages of the educational process must correspond to the stages of the formation of skills related to learning activity. Based on this point of view, Episheva constructs a general conception of a methodological system of education, allocating a place in it to preparing teachers who approach teaching in accordance with this conception.

We conclude this section with the dissertation of Smykovskaya (2002), which is devoted no longer directly to students, but to the work of the teacher. More precisely, she studies the development of the *teacher's methodological system*, which includes the teacher's aims, methodological style, and organizational formats. The formation of such a system is a multistage process, which begins during the first years of study at a pedagogical college and continues for the duration of the teacher's pedagogical career, including such stages as grasping the achievements of other teachers who are masters of the pedagogical art, forming a methodological toolkit, defining problematic points in the functioning of the system, remapping the system when encountering changes in conditions for its implementation, and so on. Smykovskaya's study is largely theoretical, but it also includes experimental work, which allows her to draw such conclusions as the following: "The type of the methodological system [developed] by the teacher depends directly on the pedagogical toolkit used by the teacher" (p. 23).

10 Teaching Aids

In surveying the studies devoted to means of instruction, including technology, it must be remembered that the revolution in the use of computers which has taken place over the last two decades was not predictable. In Volovich's (1991) work, we read that "it is mainly printed and audio teaching aids that can become widely used in the foreseeable future" (p. 28). Of course, no one today is likely to agree with this statement. Nonetheless, discussion of works that are obsolete from a technological point of view can also be useful, since psychological–pedagogical and methodological ideas do not age so quickly.

Volovich (1991) aims at "raising the effectiveness of instruction … by promoting pedagogical technologies that facilitate the algorithmization of students' learning activities" (p. 7). He sees this approach as a continuation of the approach of Vygotsky, Leontiev, and Galperin, contrasting it with so-called *associationist* psychology. Volovich's objective is "to determine which specific actions on the part of students are adequate [for assimilating computational rules and proving theorems] and to establish mechanisms of assimilation that are open to psychologists" (p. 15). For example, he claims to have found algorithmic activities that students must perform while searching for proofs for theorems and solving problems (deriving consequences from conditions, for example, is alleged to be such an activity). Consequently, everything can be reduced to teaching students to carry out the appropriate algorithms, which is the purpose of the teaching aids which he recommends. And, first and foremost, he recommends the so-called print-based notebooks (i.e. printed texts in which space is left to be filled in by students independently), whose methodological underpinnings and successful application he discusses in his conclusion.

Levitas (1991), Volovich's coauthor on many papers, has conducted a study close to Volovich's studies in its general theoretical underpinnings. However, he is concerned to a greater extent with more concrete questions pertaining to the nomenclature of teaching aids, their functions, and the methodology of working with them. He elaborates on these themes by analyzing the teaching process in schools. For example, for the formation of mental actions that students must perform in each concrete case, the students must receive a so-called *orientational* basis for action (in other words, information about new material and assignments that give them the impetus for action). From this, Levitas concludes that it is necessary to develop teaching aids capable of conveying such information to the entire class or individually. The teaching aids that he developed include screen-based, audio, and special devices and models (including testing devices), among many others. The means of instruction developed by Levitas and Volovich were widely used in the USSR.

Konkol (1998) contrasts classic teaching aids, mentioned above, with modern technological teaching aids — first and foremost, computers and graphing calculators, which are the focus of his work (completed in Poland). Noting that classic teaching aids give only a finished product, while modern teaching aids make it possible to observe the process of its creation and to experiment, he discusses the methodology of their use in the formation of mathematical concepts in students' minds, in the formation of their ability to engage in mathematical reasoning, in the formation of their ability to solve problems, and in the formation of the students' mathematical language. He analyzes various examples of useful assignments, as well as the methodology of their selection and use.

The goal of Pozdnyakov's (1998) study is to develop a conception of the informational environment of the mathematics education process and to give a theoretical foundation to the principal developments in mathematics education technology. Relying on an analysis and classification of the existing forms of representation of mathematical knowledge and on the classification of educational environments associated with the mathematics education process, he developed a two-tier computer-based technology for modeling the traditional pedagogical teaching aids in mathematics, as well as a two-parameter model of the interaction between teacher and student within the framework of an informational education environment; the parameters are (1) the means of representing mathematical knowledge and (2) the teacher's educational paradigm. The results of Pozdnyakov's study facilitate the identification of possibilities for the efficacious use of computers in mathematics education. His dissertation was based on his practical and theoretical work, which he described in numerous publications, including a monograph, methodological recommendations, and collections of interactive problems.

A recent work by Ragulina (2008), which reflects her experience in preparing a large number of teaching manuals, is devoted to the role of computer technologies in education and to the corresponding preparation of teachers of mathematics. After concluding theoretically that the paradigm of subject-based activity has undergone a transformation in the new information society, she proceeds to describe her vision of the content of the informational–mathematical and methodological–technological competence of teachers with a physics–mathematics orientation. She also develops an educational methodology that is, in her view, indispensable for the formation of such competence. In addition, Ragulina offers testing–measuring materials for identifying competence. She then cites the results of such diagnostic testing in support of the theoretical model which she has constructed, arguing that the methodological system which she has formulated facilitates improvements in the quality of teacher preparation.

11 Teaching in Elementary Schools

A number of studies have been devoted to elementary mathematics education. Practically all of them use the principles and approaches of developmental education. Activity theory (Leontiev) is also one of the main theoretical foundations of these studies. Nearly all of these studies use or cite the works of Russian psychologists — Vygotsky, Elkonin, Davydov, Krutetskii, Talyzina — as well as foreign mathematics educators and psychologists such as Freudenthal and Piaget. The methodologies of these studies also display considerable similarities: all studies include a theoretical analysis of the existing literature, and in practically all studies, pedagogical experiments constitute the most important source of data for analysis. These data include artifacts collected in the course of pedagogical experiments and interviews with teachers involved in the experiments. Some of the studies carry out analyses of school textbooks in elementary mathematics. Observations of teachers' and students' activities, as well as test results, are also commonly used. Most of the studies apply statistical methods to data analysis.

We begin our survey of dissertations about elementary education with a study by Efimov (2005), which is devoted to the question of the "human-oriented" component in the course in mathematics. Studies with similar subject matter have already been described above; Efimov, however, while reserving a place for the general analysis of such concepts as humanism and its development, focuses specifically on elementary schools, seeking to formulate a "methodology for the implementation of a "human-oriented" component in education aimed at supporting the individual development of the child" (p. 8). This objective must be achieved, in the author's view, through a "humanitarization" of the substantive–informational component and a "humanitarization" of the education process. Thus, Efimov develops certain special technologies, such as the *technology of constructing the lesson as a narrative*, which makes it possible to reproduce "the world of childhood in everyday situations, in fairy tales with favorite characters, and so on" (p. 26). During a two-year teaching experiment, he identified a number of relevant criteria — for example, a criterion pertaining to *the student's side*, which includes data concerning students' knowledge, capacity for work, susceptibility to fatigue, and other factors — that were assessed in the course of the experiment. The author claims that the experiment confirmed the validity of his approach.

Khanish's (1998) study, which relies on Polish material, is devoted to a more concrete issue, although one that has important theoretical ramifications: the development of creative abilities in the youngest schoolchildren. The foundation of her approach is problem solving. Using mathematical assignments, she seeks to arouse *surprise* in the students, which she regards as an important stage in the development of a creative approach. Her work includes what may be described as a classification of problems, as well as a discussion on the methodology behind their construction and use. The approach developed in Khanish's study was used as a basis for programs in a number of Polish schools; in this way, as she contends, the approach received pedagogical validation.

Gzhesyak's (2002) work, which also draws on Polish material, is devoted to a relatively similar issue: teaching with the use of a so-called system of *goal-oriented problems.* The construction of such a system, i.e. a system that corresponds to given pedagogical aims and thus takes into account the heterogeneity of different classes, involves a theoretical analysis of the principles of teaching children mathematics in elementary school. The author carries out such an analysis; in particular, he proposes a three-part model of a system of goal-oriented problems, which takes into account (1) the format in which knowledge acquisition is organized (individual, group, and whole-class), (2) the level of educational activity, and (3) the type of problems offered (play, standard, test, methodological) (p. 16). He likewise discusses the technology of using such systems in teaching. All of these general approaches were implemented in the actual preparation of various types of pedagogical materials and the preparation of teachers for working with them. Tests conducted in classes which employed the experimental program indicated increased effectiveness in teaching, according to the author.

Mathematical development begins, of course, before school (even elementary school). Kozlova (2003) analyzes the formation of elementary conceptions in preschool children. The course of this formation is determined by many factors, including the teachers' level of preparation. Consequently, considerable attention is devoted to this factor in the dissertation. Kozlova's dissertation research reflects her experience in writing a series of books for preschool children (the titles of which may be translated as *Mathematics for Preschoolers*, *Smarty Pants*, *Baby Square*, etc.). "For the foundation of the child's scientific development, [the author] provides a system of related small intellectual problems aimed at the formation … of certain intellectual abilities and skills" (p. 35). Kozlova devotes considerable attention to the basic concepts of set theory. Visualization and intellectual activation are presented in the work as central principles for children's intellectual development in the field of mathematics and for their teachers' professional development. Kozlova makes wide-ranging and systematic use of the history of mathematics teaching and of foreign experience.

Beloshistaya (2004) also analyzes teaching preschoolers (and young schoolchildren). Pointing out the existence of many contradictions in the mathematics education of children and, in particular, the lack of continuity and succession in it, she makes a general argument to the effect that the central goal of teaching should not be the accumulation of knowledge, but rather the students' mathematical development — understood first and foremost as the formation of a specific style of thinking. She regards modeling as the principal methodological means for mathematical development; her view is that modeling must be widely employed both in elementary school and with preschoolers. The system of teaching that Beloshistaya proposes has been employed in a number of schools and kindergartens; she compares test results from these and control classes to argue for the success of her model.

Golikov (2008) likewise studies the development of mathematical thinking in young schoolchildren and is concerned with the problem of providing for continuity in education. In his study, he undertakes an analysis of the very notion of mathematical thinking, distinguishing six different approaches to it. He examines the specific characteristics of the mathematical thinking of young schoolchildren, citing data to show, for example, that while geometry problems do not exceed 14% of the total number of problems in elementary school textbooks, in grades 7–9 their share constitutes over 40% (he sees this as one reason for the difficulty that students have with geometry in these grades). Golikov also inquires into the influence that a teacher's pedagogical abilities have on students' mathematical development. He favors the use of dynamic games, which he regards as an important means of developing thinking; his dissertation devotes considerable attention to these games and their use. According to data cited by the author, experiments conducted in schools and colleges in Yakutia have supported his ideas.

In concluding this section, let us consider two more studies whose titles contain the words "developmental education." Istomina-Kastrovskaya (1995), the author of a widely used elementary school textbook as well as a textbook for future elementary school teachers, has generalized her work in her dissertation. Actual practice has thus clearly confirmed the possibility of teaching in accordance with the principles which she enunciates. The subject of Istomina-Kastrovskaya's defense was a methodological conception and model of a developmental education system, along with related approaches to teacher preparation. Among the features of this conception, we would mention the emphasis placed on the "necessity of goal-directed and continuous formation of mental activity techniques in young schoolchildren: analysis and synthesis, comparison, classification, analogy, and generalization in the process of assimilating mathematical content" (p. 17). The study describes the contents of a course which Istomina-Kastrovskaya proposes, along with a system for organizing the educational activity of young schoolchildren.

The work of Alexandrova (2006) similarly belongs to the author of numerous popular textbooks, which are based on the ideas of D. Elkonin and V. Davydov. In line with their theories, Alexandrova devotes considerable attention to the concept of magnitude as a key feature in the study of numbers. Her dissertation also analyzes concrete methodological issues pertaining to the study of a series of other topics such as solving word problems and studying geometric material. Experimental trials of her textbooks were conducted in different cities around the country. The learning activity metrics that she cites indicate that a noticeable improvement in the effectiveness of teaching was observed in experimental classes. To cite just one parameter, if pretests administered in experimental classes showed a score of 54% for "completeness of knowledge assimilation," and a score of 55% when administered in control classes, then following the controlled experiment, they showed a score of 91% and 65%, respectively (p. 29).

12 On Teaching Specific Mathematical Subjects in Schools

The development and analysis of new mathematics courses for schoolchildren, as well as the reorganization of existing courses, are reflected in scholarly works and, in particular, in dissertations. Among the studies devoted to new approaches to teaching geometry, we cite the work of Podkhodova (1999). The distinctive characteristic of this study consists in the fact that, while traditionally the systematic course in geometry started in seventh grade (sixth grade in the old numeration), Podkhodova sets up the task of developing a systematic course for grades 1–6, i.e. of viewing the informal study of geometry as a unified course and enriching it with new material. Consequently, the aim of her research is to provide a theoretical–methodological foundation for the construction of such a course, taking into account new conceptions of education. Among the key principles identified by her are cohesiveness and unity, the simultaneous study of plane-geometric and three-dimensional objects, and paying attention to the students' subjective experience. She underscores the importance of making a special selection of educational materials and constructing a special system of problems. Her study makes substantial use of psychological research. She has prepared numerous teaching manuals on the basis of the theoretical propositions articulated in her study.

Orlov's (2000) work is based on ideas that are similar to the work just discussed. The aim of this researcher is to construct a course in geometry for ordinary schools, not as a course that conveys an already-existing body of knowledge, but as a course that relies on active cognitive activity by the students and on their experience (pp. 4–5). Consequently, the study contains both an analysis of various approaches to teaching geometry and a discussion about works on child development. Subsequently, the author turns to the theoretical principles on which the course he envisions must be based (such as the requirement that the material studied be organized in large blocks, that two-dimensional and three-dimensional geometry be studied simultaneously, and that various types of independent work be included in the course). In his conclusion, Orlov describes the results of experimental teaching which, according to him, confirm the positive influence of his methodology on the development of students' intellectual abilities.

Totsky's work (1993), which draws on Polish material, proposes constructing a course in geometry on the basis of what the author calls a "locally deductive approach." According to him, such an approach involves the creation of "little deductive islands" — minisystems linked up into thematic lines — and also gives a prominent role to inductive reasoning, with only a gradual generalization of concepts and properties (p. 20). He describes a corresponding methodology and cites experimental teaching data and its results.

While the two studies just cited are devoted to seeking new ways of teaching a traditional course, the work of Ermak (2005) goes substantially beyond the bounds of traditional organization: its subject is the construction of an integrated course in geometry and natural science. Therefore, although the structure of this study resembles that of the studies described above (analysis of existing work–theoretical construction–experiment), its approach differs because of its extensive use of nonmathematical and nonmethodological literature. In particular, the author draws the conclusion that one of the reasons for students' difficulties stems from an unjustifiable lack of attention to what she calls "the psychological structure of geometric images," noting "the paucity of individual aggregates of spatial figures, geometric representations" (pp. 25–26).

In this section, we should also mention the work of Tazhiev (1998), although its title, "A Statistical Study of School Education as the Basis for Didactic Models of Mathematics Education," might lead one to believe that its main content concerns a statistical study of school education (in Uzbekistan). In reality, only one chapter of this study is devoted to these disheartening statistics, while the rest deal precisely with the teaching of geometry. The author conducts an analysis of the concepts studied in the course in plane geometry (determining, for example, that the course is overloaded), discusses the pedagogical foundations of teaching proofs, proposes a didactic model for increasing knowledge in three-dimensional geometry (pointing out the utility of solving problems and using visual models), and finally addresses the importance of a practical orientation in education. He reports on experiments that he has conducted, but does not discuss them in his summary.

Breitigam (2004) studies the problem of students' comprehension and assimilation of elementary calculus. This leads her to ask what precisely is meant by "comprehension" or "meaning." Unpacking the significance of these words, the author offers several definitions, including a logical–semiotic definition, a personality-based definition, and a structural subject-based definition, which consists of identifying the basic idea in a concept and establishing a substantive connection between ideas (pp. 8–9). She characterizes the approach which she developed as "pragmatically semantic" and involves the identification of basic educational ideas and objects, as well as the use of various forms of representation of knowledge. She emphasizes the selection of problems and laboratory projects, as well as dialogs "aimed at getting the students to grasp various contexts of meaning" (p. 31).

The work of Sidorov (1994), in contrast with almost all of the studies described in this section thus far, contains no description of an experiment; it is based on the author's numerous publications and represents their theoretical generalization. He sees his novel contribution in this study as consisting in "the development of theoretical approaches to the creation of courses in mathematics for secondary schools and colleges on the basis of a conception of their continuity" (p. 5). He distinguishes between three levels of requirements for students — minimal, middle, and heightened — and proposes constructing school courses in such a way that they might include a certain core (mandatory topics) as well as "outer layers" added in accordance with plans for the students' future studies and also in accordance with the preferences (likely more subjective ones) of the teacher and the student. A coauthor of numerous textbooks, Sidorov then goes on to demonstrate how these textbooks correspond to the theoretical views that he has laid out.

Plotsky's (1992) dissertation, which makes use of Polish material, is devoted to stochastics, a new field for Russian (and Polish) schools. Like the previous study, this dissertation is based on the author's numerous textbooks and aims to give a foundation to the notion of teaching stochastics within the framework of "mathematics for everybody." The author's model involves (1) active instruction in mathematics (mathematical activity), (2) the study of stochastics as a body of student-discovered methods for analyzing and describing reality, (3) the study of problem situations as sources of stochastic problems, and (4) the use of inactive and iconic means of representing stochastic knowledge. He buttresses his views and conclusions with references to his textbooks.

13 Teaching in Nonpedagogical Institutions of Higher Education

Several doctoral dissertations have been devoted to mathematics education in technological universities or in the mathematics departments of (nonpedagogical) universities. Often, the works focus on new approaches to teaching existing courses or on developing new courses, including courses that appear to the authors to be useful for the "humanitarization" of education. Usually, a study involves the development of new teaching manuals. The pedagogical experiment is the main research methodology applied in these studies, along with observations, interviewing, questioning, and testing.

Beklemishev's (1994) study can serve as an example of work with sufficiently traditional courses. He has designed a course and a corresponding textbook that integrates analytic geometry and linear algebra for university students studying physics and mathematics, or physics and engineering. The combination offered by the author provides significant convenience, as his own practical observations confirm, enabling improved and faster assimilation of necessary knowledge.

The work of Sekovanov (2002) is devoted to a subject that is new to colleges: fractal geometry. The author examines this subject as a means of developing students' creativity. Pointing out a number of problems and contradictions in contemporary higher education (such as the gap between the need for creative professionals and the reproductive nature of educational processes in many universities), he proposes a program for studying fractal geometry, which in his opinion facilitates the development of student creativity. Consequently, his work analyzes the theoretical aspects of the problem and also offers practical recommendations, which are embodied in a series of manuals written by him and tested out in actual teaching.

Perminov (2007) examines the problems of studying discrete mathematics in secondary schools and universities and providing for continuity in this branch of study. He points out a gap between the secondary school requirements in discrete mathematics and the way in which discrete mathematics is actually taught in universities; he suggests that continuity in the study of discrete mathematics might be strengthened if this study were conducted within a computer science framework and particularly if emphasis were placed on the role of discrete mathematics as "a foundation for teaching [students] to design a complete sequence of steps in the use of computers" (p. 6). The author has developed an overall conception of the continuous study of discrete mathematics; within the framework of this conception, he has developed instructional materials for schools and universities that have been used in various locations.

The work of Kornilov (2008) is devoted to the teaching of rather specialized issues in applied mathematics, but he examines these issues in light of the changing approaches to university education that are collectively referred to as "humanitarization." Consequently, after discussing the general theoretical aspects of this notion, he seeks to define the humanities-oriented component of the course topic that is the focus of his study (for example, he explores the means that the course might offer for the students' personal growth). Kornilov has developed various methodological recommendations, which in his view represent the practical importance of his results.

Gusak's (2003) dissertation is based on many years of experience in using the textbook that he has written for natural science majors at universities; this textbook has gone through multiple editions. (He claims that this is one of the most stable mathematics textbooks for university students who are not majoring in mathematics.) His work includes a theoretical examination of the pedagogical effectiveness of textbooks and is based on an analysis of the pedagogical and methodological literature and mathematical programs in university education. He cites a pedagogical experiment that took place over 32 years (1971–2003) and was accompanied by observations, questionnaires for university students and teachers, and the evaluation of students' mathematical knowledge acquired by working with specially designed instructional materials. Gusak emphasizes that a textbook's structure must possess several fundamental characteristics, including purposefulness, a systematic approach, the sequential presentation of educational material, logical and semantic unity, and openness. The didactic principles which he recommends applying when designing instructional materials and university textbooks include visualization, simplicity and clarity, the nonformal introduction of mathematical concepts, the verbal interpretation of formulas, explicit connections between different chapters and sections of the textbook, the integration of the mathematical course with the specific scientific context that is relevant to the students, the inclusion in the textbook of the theory of computational methods, and integration into the history of mathematics.

Rozanova (2003) also devotes her study to teaching students who are not mathematicians, but she is more occupied with a general problem: that of cultivating mathematical literacy among students at technological universities. This topic involves her in defining what constitutes mathematical literacy, analyzing the history of the development of modern mathematical literacy, and searching for practical ways to raise students' mathematical literacy. She views the mathematical literacy of future engineers as a system of mathematical knowledge and skills that is applicable to their professional, sociocultural, and political activities, and leads to the fulfillment of their humanistic and intellectual potential. She claims that the mathematical literacy of the graduates of a technological university is formed when their mathematical reasoning is developed, when they become aware of the importance of mathematics as science, and when they are able to use the mathematics that they learned at the university in their professional lives. She considers the main result of her study to consist of her formulation of the conception of the development of mathematical literacy among students of technological universities, and the development of a methodological model through which this conception may be implemented in actual practice.

The work of Salekhova (2007) was carried out in a pedagogical college and thus belongs to a category of studies which will be mainly addressed below, but we will analyze it in this section because the preparation of mathematics teachers does not constitute her main content and, as she remarks, "the model designed here may be implemented not only in pedagogical colleges but also in other educational institutions" (p. 16). This dissertation is devoted to developing approaches to teaching mathematics in English to students who study in special groups with an advanced course in the English language (note that the expression "bilingual education," frequently employed by the author, may be misleading: the dissertation is concerned with the study of a foreign language). Salekhova asserts that teaching mathematics in English facilitates the development and advancement of the linguistic and mathematical competences of the students and promotes their ability to seek out and productively receive mathematical information in two languages. She points out that while the need for such a course is apparent, no experience in such teaching exists in practice. According to her, the experiments she has conducted here have confirmed the efficacy of her approach.

14 Mathematics Teacher Education

Probably the largest category of studies consists of works devoted to teacher preparation. We have already touched on this topic in earlier sections of this review, but now we will turn to those dissertations whose content is mainly, if not entirely, connected with education in pedagogical universities [recall that such specialized universities existed in the USSR and continue to exist in Russia today (Stefanova, 2010)].

The topics of these studies vary and include implementing modern educational principles in teacher education, reforming teacher education, preparing teachers to teach specific topics and subjects in mathematics, preparing teachers for schools with an advanced course of study in mathematics, and developing teachers' proficiency in advancing students' mathematical creativity as well as developing teachers' own mathematical creativity. As with many studies already examined, these works are rooted in the theories of Vygotsky, Leontiev, Davydov, Zankov, Elkonin, and other important Russian psychologists. Problem solving is typically treated as a major form of learning mathematics and, therefore, works — theoretical or practical — about problem solving (Boltyansky, Vilenkin, Gusev, and Sharygin, as well as Polya and Freudenthal) also serve as a source for many studies.

Methodologically, the studies examined in this section probably do not differ on the whole from the studies examined above. Many authors of these studies have designed mathematical or educational courses for prospective mathematics teachers and provided textbooks and other instructional materials to support these courses. Many of the studies include evaluations of the effectiveness of these courses and the educational ideas on which these courses are based. Consequently, many of the studies are aimed at theoretically interpreting what has been done and verifying it experimentally. Many also use such research tools as direct and indirect observations, interviews, questionnaires, analytical conversations, and analysis of students' and teachers' achievements.

14.1 *General Questions of Mathematics Teacher Education*

Russian scholars regard the preparation of mathematics teachers as a unified process, grounded in unitary principles and approaches, and implemented through the development of teaching plans (programs) as well as textbooks and other instructional materials. The development of such a unified didactic system of mathematics education for future teachers is the aim of Smirnov's (1998) dissertation. The author identifies and describes the principal components of such a didactic system, which include: Motives; Goals and Objectives; a Model of the Content and Structure of Mathematics Education; Means, Forms, and Conditions; and Results and Testing the Functioning of the System (p. 22). He offers a special educational technology, which he characterizes as "visual/model-based." In his concluding chapter, Smirnov describes an experiment he conducted, as well as his criteria for judging the effectiveness of the teaching process. The table below,

Table 1. Parameters of assimilation of the topic as a coherent whole.

Parameters of assimilation of the topic as a coherent whole	E (average)	C (average)
Knowledge of the basic components of the topic	3.09	2.31
Knowledge of the structure of internal connections	3.58	2.46
Understanding of the structure of external connections	2.94	2.44
Degree of integration	2.98	2.48
Functionality	3.47	2.45
Generality	3.06	2.22

for example, indicates scores reflecting the degree to which the topic "Elementary Functions" has been assimilated as a coherent whole by experimental (E) and control (C) groups of students (p. 32).

Stefanova's study (1996) is devoted to the methodological preparation of future mathematics teachers. She also considers the system of such preparation as a coherent whole. Along with traditional components (such as the goals, content, and methods of instruction) she emphasizes the expected results of instruction, "which are of a highly personalized nature" (p. 12). The importance of a personalized approach in general is emphasized in every possible way by her [note that among the goals of methodological preparation, along with competence and professionalism, she also lists individuality (p. 26)]. Stefanova proposes a model for the content of teacher preparation and a model for testing out the functioning of the system. Among her contributions are the development of programs for a series of courses, which have been successfully taught over a number of years, and a textbook that is used at various pedagogical universities.

While Stefanova studies questions connected with the entire system of the methodological preparation of teachers (which involves the methodological interpretation of mathematical knowledge, the importance of which she stresses and which may in principle occur in various different courses), Liubicheva's (2000) dissertation, which relies on Stefanova's work, is devoted to issues connected specifically with planning a course on the methodology of teaching mathematics. The author's conception devotes particular attention to planning teaching activity, to her own teaching of future teachers, to the formation of teachers as "subjects who direct the pedagogical process" (p. 7), and to the development of mathematical communication abilities. She developed a new program for the entire course on the methodology of teaching mathematics (which lasts several semesters).

The work of Kuchugurova (2002), carried out under the scientific influence of Smirnov, has as its aim the "theoretical and practical grounding of an innovative model of the process of the formation of the future mathematics teacher's professional–methodological abilities" (p. 6). The researcher emphasizes the importance of a systematic and unified approach, on the one hand, and the problem of a personalized orientation in instruction on the other hand. She proposes a model of the pedagogical process in which the development of the subjects of instruction (in other words, students) is represented in the form of a spiral. Considerable attention is devoted to systems of assignments that "make it possible to put the student into a situation that offers the possibility of developing [educational] activity." The requirements for such assignments are formulated and substantiated. Experimental work based on the use of the author's methodological approaches and recommendations has, according to her, confirmed their legitimacy and effectiveness.

The dissertation of Zlotsky (2001), defended in Uzbekistan, is devoted to the system of mathematics teacher preparation in the context of general university education (although it may be supposed that a large, even if not the largest, part of the graduates from the university whose material formed the basis of this study were being prepared specifically for future teaching in schools). The researcher consequently analyzes both the mathematical and the pedagogical components of the teacher preparation system. His study emphasizes the necessity of imbuing teachers with mathematical literacy, which in turn is also useful for students who will not go on to become teachers (as an example of such a "dual action" topic, he cites the Frobenius theorem, which effectively demonstrates that no other "good" number systems exist besides the ones studied in school). Zlotsky also discusses the importance of mathematical modeling (he has developed related courses) and methodological abilities and skills. Control assignments and psychological tests were used to assess the effectiveness of the proposed system.

A recent work by Sadovnikov (2007) addresses the methodological preparation of teachers in the context of the "fundamentalization" of education. As far as it is possible to judge, the term "fundamentalization," employed by the author, is of comparatively recent origin. The author uses it to refer to such phenomena as "the identification … of essential knowledge, the integration of education and science, the formation of a general cultural foundation in the process of education" (p. 13). In accordance with this description, he identifies requirements for teacher preparation in the context of fundamentalization. For example, he proposes "teaching future mathematics teachers how to form the basic structural units of mathematical knowledge in the minds of the students" (p. 17). As for the integration of science and education, according to him that is achieved when the content of education reflects, as far as possible, "the currently corresponding content of science" (p. 33). The author has developed certain specialized courses, particularly courses devoted to the logic and role of problems in the school course in mathematics, which, again, according to him, have undergone successful trials.

Questions concerning the continuity and multistage nature of education, which have been mentioned above in relation to school education, are studied at the college level as well. They are the subject of the work of Abramov (2001), which analyzes the functioning of, and connections between, a three-year teacher training college and a pedagogical university. To assess the connections between the stages in a teacher's preparation, the author proposes a special mathematical function, "whose values reflect the effectiveness of the assimilation of the subject-specific, professional content of instruction" (p. 9). In general, the work makes extensive use of mathematical techniques; for example, "a set of didactic units of educational material" is defined "using graphs of dependence and matrices of logical connections" (p. 9). Although we cannot provide a complete description of the special mathematical function referred to above, we should note that its values depend in turn on eight parameters, which might be difficult to define in practice. Experimental work aimed at implementing Abramov's system of teacher preparation was conducted over a number of years and, according to him, met with success, with many concrete programs and didactic materials being developed during the course of the experiment.

The work of Malova (2007) goes beyond the framework of teaching in a pedagogical university, since its subject is the continuity of the methodological preparation of the teacher as a whole. Malova analyzes the problem from the perspective of so-called *subjective coherence*; from this perspective, the "teacher's continuous methodological preparation is a process that involves the formation of the pedagogue as the subject of his own methodological development" (p. 11). Of the study's four chapters, the first is devoted to methodological issues (including the importance of overcoming stereotypes that teachers develop); the second addresses the problem's theoretical aspects; the third describes recommended ways of providing for the continuity of methodological preparation; and, finally, the fourth discusses experimental work in a pedagogical university and professional development institutions.

In concluding this section, let us touch on one more general problem which was mentioned earlier in connection with the topic of education as whole, but which is also studied specifically in the context of teacher education. Naziev (2000) has studied questions related to the "humanitarization" of mathematics teacher preparation. As he writes, at the present time, "the center of gravity in school education is shifting from studying mathematics to educating with mathematics" (p. 7). He regards teaching students how to search for proofs as a crucial means of "humanitarizing" the teaching of mathematics. Consequently, after arguing that mathematics means proofs and that the teaching of mathematics means spurring students to discover their own proofs, he concludes that the teaching of mathematics constitutes an irreplaceable means of ethical education and of instruction in "the science of human freedom" (p. 17). As a result, he considers it necessary, as a supplement to courses in algebra, geometry, and so on, to establish courses of a general mathematical character in pedagogical universities, which would generalize and systematize what has been learned and which would possess humanities-oriented potential in the sense described above. Naziev's study generalizes his experience in teaching such courses.

14.2 *Special Aspects of the Methodological Preparation of Future Teachers*

While the dissertations discussed above are devoted to designing the methodological (professional) preparation of teachers as a whole, a number of studies deal with separate aspects of such preparation. The aim of Perevoschikova's (2000) work is "to develop a theoretical–methodological foundation for the preparation of the future mathematics teacher for diagnostic activity" (p. 11). Consequently, the author examines such problems as the future teachers' integration of diagnostic knowledge obtained from different disciplines and the formation of their own diagnostic abilities. She stresses the need to change the traditional system of testing, since, in accordance with the new goals of education, the focus must be not only on testing the assimilation of specific knowledge but also on testing students' command of various methods of activity, and even on assessing the record of the students' emotional–axiological attitude toward learning. Perevoschikova developed a theoretical model of such diagnostic activity, singling out its various structural components (motives, goals, objects, means, etc.). In particular, one chapter of the study is largely devoted to developing a diagnostic toolkit. The author conducted an experiment with teaching a course on "Methodological Issues in Diagnostics," which resulted in noticeable growth in the diagnostic abilities of the experimental groups compared with the control groups.

Several studies are devoted to the development of creativity in future teachers and/or future students of future teachers. The goal of Afanasiev's (1997) work is to develop and justify principles and corresponding instructional tools aimed at the development of creative activity by prospective teachers in the process of problem solving. The author sees the principal means for the formation of such activity as consisting of a body of educational–methodological problems, and this means will be effective, in his opinion, if "the problems may be solved using nontraditional methods" (p. 7). In his analysis of creative activity, Afanasiev relies extensively on the existing literature on problem solving. One of his contributions, as he writes, is "to develop an algorithm for pedagogical actions aimed at solving new, original problems, designed by us, which constitute a nonstandard system of knowledge" (p. 34). His theoretical approach has been embodied in a course that he has developed and taught: "Theory of Probability and Mathematical Statistics." Of note is the assessment system which he chose to evaluate the efficacy of his approach. He established certain parameters of student activity (such as the frequency of modeling or the frequency of using various solutions or simply the average score), and then evaluated students' performance along these parameters, not only in his own class but also in other, concurrently offered courses in mathematics. By the end of the course that he himself taught, Afanasiev recorded a definite improvement in student activity along the parameters he selected. In addition, it turned out that the quality of the students' assimilation of the content of this course (defined, again, in accordance with the methodology developed by the author) had also improved.

The work of Dorofeev (2000) is similar to the study just described. Its aim is "to develop a foundation for the theory and practice of the formation of the creative activity of future mathematics teachers ... by means of teaching them to search for rational solutions to problems" (p. 7). The author proposes a new approach to teacher preparation based on "a system of interconnected school-level geometric problems, mathematics exercises, and simulation exercises, which facilitate the formation of the student's ability to 'make discoveries' " (p. 11). He defines four levels in the development of creative activity and offers an instrument (a set of problems) for determining the level attained by a teacher; he also offers certain methods and means for raising teachers to higher levels, which are contained in the manuals he has written and the courses he has designed. According to him, during the final assessment, over 70% of students in experimental groups, for example, solved the problems given to them, while only 50% of students in control groups solved these problems. These and similar metrics enable the author to argue for the effectiveness of the approach he proposes.

In contrast with the two studies just described, Ammosova (2000) is concerned not so much with the problem of developing the future teacher's creative potential as with preparing the teacher to develop the creative potential of the students, specifically elementary school students. To develop the elementary school student as a creative personality, in the author's view, means to (1) help the student acquire creative abilities, (2) develop the student's creative imagination and intuition, and (3) stimulate the student's activity by placing demands on the student (p. 20). On the basis of the theoretical conception she developed, Ammosova has prepared the requisite methodological supporting materials: courses in mathematics for future elementary school teachers; programs and special courses for them, including courses that prepare them for teaching electives to schoolchildren; and a system for teaching future teachers principles for selecting bodies of problems for elementary school students.

Since the mid-1980s, the importance of differentiated mathematics education has received increasingly great emphasis; therefore, questions have arisen about how to prepare teachers for a new education system that includes classes of different levels, and in particular, questions about preparing teachers who are capable of teaching in classes with an advanced course of study in mathematics. Ivanov's (1997) work grew out of his experience in organizing and teaching a pedagogical major at the St. Petersburg University's mathematics department, oriented toward preparing highly educated mathematics teachers. The aim of his study is "to identify opportunities for combining the fundamental and research–scientific preparation of student–pedagogues in the mathematics departments of classic [nonpedagogical] universities with their professional preparation as future teachers in specialized schools" (p. 2). The author formulates and offers supporting arguments for several theoretical principles that he follows in his methodological designs. Among them is the principle that education is cumulative, according to which relatively small quantities of acquired information at certain stages may produce structural changes in the system of knowledge and intellectual development; or the principle that education is polyphonic, according to which it is possible to organize the education process in a way that integrates various content-methodological lines; and so on. The author introduces the notion of a cluster of concepts, propositions, and problems, with which he explains how many interconnected concepts may be discussed that are related to the same topic. Ivanov's theoretical work generalizes his practical work, which has included writing a course in school mathematics that emphasizes investigating numerous connections and parallels with more advanced courses.

The work of Petrova (1999) is also concerned with the problem of preparing teachers for specialized mathematical schools (and even, as she herself writes, schools for the mathematical elite), but she is more focused on the pedagogical and methodological sides of this preparation. For example, in her view, this preparation must include special courses with in-depth study of school-level mathematics, as well as courses that integrate psychology, general pedagogy, and the methodology of mathematics teaching. The system developed by Petrova (1999) also requires students to write final theses on topics connected with in-depth instruction in mathematics. Her theory about how a system for student preparation should be designed, on which all of these proposals are based, relies on numerous studies of system-based approaches in general and in pedagogy in particular. She also offers her own diagnostic system for assessing whether future teachers have reached the requisite level; this system has shown that students who have been prepared within the framework of the system proposed by her are better prepared to teach an in-depth course than ordinary students.

Drobysheva (2001) is concerned with the broader issue of preparing teachers who are capable of implementing differentiated education. She notes that, at present, all teachers must be able to conduct differentiated education, i.e. education that takes individual abilities into account; meanwhile, neither the theoretical nor the practical side of such preparation has been systematically worked out. She posits that students' distinctive individual characteristics may be of two types: the first type consists of characteristics which, in her opinion, can be taken into account without adjusting the content (characteristics connected with attention, temperament, character, etc.), while the second type consists of those characteristics which it is impossible to take into account without recoding the content (different types of perception, different types of memory, different forms of reasoning, etc.); thus, the teacher must be prepared for such recoding. Drawing on an analysis of existing literature, Drobysheva describes the components that such teacher preparation must include, specifying that it is not enough, for example, to offer a list of studied topics, but that it is necessary in some measure to describe the relevant body of educational materials. Her theoretical program has been embodied in a monograph which she has written, as well as in concrete materials and courses she has developed. She has also developed diagnostic materials, which show that if prior to experimental teaching in a given group *not one* student possessed the skills required to carry out differentiated work, then after experimental teaching such skills were possessed by practically all students (except the very few who had to be dismissed from the group anyway).

In concluding this section, we should mention the dissertation of Silaev (1997), which is devoted to preparing teachers to teach the school course in geometry. The author offers his own idea of how such preparation may be improved, "based on an understanding of such preparation as a synthesis of preparations in courses in geometry, elementary geometry, and mathematics teaching methodology" (p. 8). He formulates the principles according to which the relevant instructional–methodological toolkit must be designed. It is noteworthy that his theoretical analysis includes an examination of foreign findings; moreover, he notes that "the teacher's ability to carry out a critical analysis ... of foreign findings concerning teacher preparation constitutes one of the factors in the improvement of methodological preparation" (p. 14). Unfortunately, it is impossible to determine from the author's summary which countries' experiences he has analyzed and which must also be analyzed by the teacher. In addition, the author investigates how cognitive techniques are formed when the teacher solves geometric problems. He offers a schema of the formation of such techniques, which involves identifying the technique's logical structure, studying its basic characteristics, determining the main types of geometric problems connected with the technique, and so on. He has embodied his theoretical ideas in a number of methodological manuals, video courses, and problem books.

14.3 *On Teaching Mathematics to Future Teachers*

In Russia (and the USSR), the profession of teacher was, and continues to be, chosen by students at the moment when they enroll in a pedagogical institute (university). At these institutions, all mathematical subjects are taught not to a general audience of students who have decided to learn some mathematics, but to individuals who plan to become teachers in the future. One may therefore inquire about the specific characteristics of teaching mathematics to such an audience. This topic is the subject of many dissertations.

Let us begin with the work of Gamidov (1992), which is devoted to the mathematical preparation of future elementary school teachers. After pointing out many shortcomings encountered in the mathematical preparation of elementary school teachers, Gamidov proposes his own theoretical and practical conception of the mathematical preparation of future teachers, which is embodied in a system of methodological recommendations. His key idea may be summed up in highly abbreviated form as an attempt to integrate strictly subject-based elements and didactic elements, always underscoring both connections with school education and pedagogical methods and technologies. The study devotes considerable attention to the history and specific character of education in the author's country, Azerbaijan.

Most studies, however, are devoted to the mathematical preparation of mathematics teachers not in elementary but in basic and senior schools. Shkerina (2000) notes that, at the time her dissertation was written, there were no systematic studies of the cognitive–educational activity of students undergoing mathematical preparation at a pedagogical university (p. 5); her objective was to fill this gap. She emphasizes that it is not enough for future teachers to acquire a specific body of knowledge themselves: "they must be prepared to organize the actual mathematical activity of their students" (p. 14). Consequently, she identifies groups of necessary actions in the mathematical activity of a student, among which are learning the definitions of mathematical concepts, identifying the basic features and properties of mathematical objects, establishing logical connections between mathematical objects in one or several mathematical theories, and carrying out actions pertaining to problem solving. On the other hand, Shkerina identifies groups of actions performed by students in the process of learning activities (for example, testing/self-testing the assimilation of knowledge or its reproduction). Drawing on this type of analysis, she proposes models and technologies of education, whose success she confirms by citing an experiment that she has conducted.

Safuanov's (2000) work is also devoted to identifying methodological principles for designing and implementing a system of instruction in the mathematical disciplines as they are taught at a pedagogical university. This author developed his conception by relying on the genetic approach, which he defines as "following the natural paths of the origin and use of mathematical knowledge" (p. 6). He undertakes a theoretical investigation of the genetic approach and its principles, in particular analyzing the works of Russian methodologists of the first half of the 20th century, who effectively promulgated the genetic method of exposition. He also discusses the works of contemporary foreign researchers, which is quite rarely done in Russian studies of mathematics education. As a practical example of the application of his general conception, he works out a three-stage system for studying the following topic at a pedagogical university: divisibility of integers — Euclidean rings — polynomials (presenting the natural development of an idea). Other concrete methodological recommendations are formulated as well. The author's proposals have been put into teaching practice at some pedagogical universities.

While the two studies just mentioned are devoted to general issues in mathematics education, the recent work by Kalinin (2009) deals exclusively with the teaching of differential and integral calculus. In this work, several themes may be identified. First, the author proposes new (at least for a pedagogical university) mathematical approaches to defining the basic concepts of calculus, along with new mathematical topics that may, according to him, facilitate the presentation of elementary calculus in schools in a manner accessible to the students. Second, the author proposes and advocates certain formats for organizing instruction, which are connected, for example, with scientific research done by the students; and methodological approaches connected, for example, with the problem of developing an in-depth understanding of the course. Third, the author offers a theoretical analysis of the requirements for teaching calculus based on the "fundamentalization" of education, which he defines as a convergence of the educational process and scientific knowledge. The ideas proposed by him were implemented over a number of years at a pedagogical university in Vyatka.

Another recent work, by Sotnikova (2009), is devoted to the organization of the activity of pedagogical university students in discovering substantive connections in the course in algebra. After noting that the knowledge of pedagogical university graduates is often lacking in depth, and in particular that they often have no understanding of the course in mathematics as a unified whole, the author gives examples of subjective connections in the course in algebra, which are, in her opinion, especially important because the course is quite abstract. Such connections, for example, may be seen in the analogies between the conceptions and propositions studied in different areas of algebra (group theory, ring theory, theory of algebras). The author offers a theoretical analysis of the notion of a "substantive connection"; she also develops a theoretical interpretation of the process by which students come to grasp the course in algebra. Her general analysis constitutes the basis of her methodological recommendations, which are aimed in particular at stimulating and organizing independent work by the students on establishing substantive connections in algebra. In connection with these ideas, the author has prepared the model (program) for a pedagogical university course in algebra, which she has put into practice.

Kostitsyn's (2001) work is devoted to teaching geometric modeling and developing the spatial imagination. He describes an experiment he conducted in which fifth-year students were given two problems from a pedagogical university entrance exam. Only 17% of the students solved the problems, while only 15% made correct representations of the objects involved in the problems (the other 2% found the correct answers using incorrect diagrams). However, when comparatively difficult problems were given with diagrams in the next experiment, 80% of the students solved them. Thus, Kostitsyn concludes that the ability to construct geometric models — and, more broadly, the spatial imagination — is actually not developed at all over the years of study at a pedagogical university. He proposes several courses meant to help in this respect, enunciating numerous concrete suggestions, some of which pertain to the use of technology. These courses have been taught at certain pedagogical institutes.

Among the studies that we are discussing, two are devoted to teaching logic at a pedagogical institute. Igoshin (2002) undertakes a multifaceted analysis of the role and place of mathematical logic and even logic in general, for example in comparison with intuition. Turning to practical issues in education, he takes the position that logic and the theory of algorithms must be taught not just as a separate mathematical subject, but as the most fundamental and leading subject which supports teaching of all other mathematical subjects. Consequently, he emphasizes the importance of identifying and clarifying for the students the use of logical principles in all mathematical and even methodological disciplines that they study. He likewise emphasizes how important it is for each component of the course to have a professional orientation — in other words, how important it is that it be directed at the students' future pedagogical activity. These ideas are embodied in courses and teaching manuals developed by him.

Timofeeva's (2006) study is devoted to designing a course in mathematical logic on the basis of the so-called theory of natural deduction. She notes that, for example, the work of Igoshin (2002), discussed above, is based on the traditional format of the course and "relies on the didactic possibilities mainly of its linguistic component, while the deductive component of the course is practically unused [in the study]" (p. 3). Preserving the content of the course, Timofeeva structures it in a different fashion, as she writes, "thus providing for the study of the most adequate, simple, and visual models of proofs" (p. 3). She contends that some of the approaches she suggests may be used directly by future teachers in schools. Her theoretical analysis is multifaceted and, in particular, includes the identification of various types of deductive activity. The study describes many concrete methodological proposals and recommendations, reflected in practice in manuals and implemented programs.

14.4 *Technology in Mathematics Teacher Education*

In this subsection, we will mention only one study, the only doctoral dissertation we possess which is entirely devoted to the use of technology in teacher education: the work of Kapustina (2001). In this work, according to its author, the computer mathematics system (to use the author's terminology) *Mathematica* is examined for the first time in Russian science as a means and basis for the creation and use of new educational information technologies (p. 10). In particular, she "identifies the role of *Mathematica* as an environment for posing and solving new subject-based and educational problems in the mathematical disciplines and describes the influence of its special objects … on the methodology of mathematics and the methodology of teaching mathematics" (pp. 10–11). She analyzes the problem of using computer systems in education and the Russian and foreign literature about this topic; she also formulates basic theoretical principles and propositions concerning the use of computer systems in education. Her proposals have found practical application in the development of a number of courses and in the writing of several monographs, methodological recommendations, and computer problem books.

15 On Candidate's Dissertations

In discussing the Doctor's dissertations defended over the past 20 years, we have mentioned if not all, then almost all, of the dissertations that have been written; for Candidate's dissertations, of course, no discussion on a similar scale is possible as many hundreds of them have been defended. Tkhamofokova and Dalinger's (1980) index contains only dissertations defended before 1980, and their number too is not small (for example, the section on the history of mathematics education alone contains 50 works). Our aim here, therefore, is very modest: only to provide a sketch of what a Candidate's dissertation looks like, without claiming that our survey is exhaustive or that the dissertations discussed in it are the best (or, conversely, the worst) of those that have been written. We limit ourselves to five studies, whose selection was essentially random, since it was determined by the selection of the most recent authors' summaries of Candidate's dissertations that happened to be in libraries at the time when we were collecting our data, and also by our wish to represent works in different areas.

Thus, let us consider the dissertation of Shagilova (2008), which is related to her study mentioned in the earlier section on "Problem Solving." Shagilova studies the changes in the role of problems in the mathematics education process and seeks to identify the factors that influence these changes. She analyzes textbooks and the statements of their authors and other educators concerning the role of problems. Two chapters of the dissertation (out of four) are devoted to recent history, from the middle of the 20th century on; she reaches the conclusion that in recent times problems have become a means of education, mental training, and development. She does not confine herself to historical analysis, but also designs specific blocks of problems (in accordance with the view of the role of problems that she develops) and puts them to use in experimental teaching.

The development of students' intellectual–creative activity is the subject of a dissertation by Lebedeva (2008). Her main tenets are support for open assignments and integrated courses, and flexibility in organizing the interaction between teacher and student. The dissertation contains two chapters, the first of which analyzes the actual notion of intellectual–creative activity and the necessary conditions for its development, while the second directly discusses the methodology of this development, presenting various assignments and considering the requirements that their design must satisfy.

Kokhuzheva's work (2008) is devoted to the formation of school graduates' preparedness to continue their mathematics education in college. The first of the dissertation's two chapters theoretically analyzes the question of what is meant by the formation of preparedness to continue education; the second describes the organizational and technological (methodological) components of recommended approaches to forming such preparedness (in particular, the author discusses the organization of elective courses). Kokhuzheva conducted experimental teaching followed by a questionnaire survey, which she cites to support the validity of her approach.

Arsentieva (2008) studies issues pertaining to the methodology of teaching algebra; her aim is "to develop a theoretical foundation and methodological support for the advanced study of algebraic structures in the school course in mathematics" (p. 4). She argues for the importance of studying such concepts as operations and groups in school; she also considers it important that students form a notion of isomorphisms. Her dissertation, like the two discussed immediately above, contains two chapters, the first of which is theoretical in nature, while the second is devoted to methodological aspects. In particular, she describes an elective course that she has developed and offers various systems of problems and exercises. She has carried out pretests and posttests which, as she notes, have confirmed her hypotheses.

Berdiugina (2009) focuses on the preparation of future teachers. The key concepts for her are geometrical abilities and techniques of learning activity. Among the first she names are logical, spatial, investigative, and other abilities. Among forms of learning activity, she lists cognitive, constructive, practical, developmental, and other activities. Berdiugina establishes connections between these concepts and focuses on the development of students' geometrical abilities on the basis of techniques of learning activity. She discusses the theoretical aspects of the problem in the first chapter of the dissertation, while in the second chapter she turns to its methodological aspects. The second chapter also discusses experimental work carried out over a number of years, during which she taught a course in geometry for first-year pedagogical institute students. According to Berdiugina, her methodology made it possible to develop the students' geometrical abilities better.

As may be easily seen even from this very brief analysis, the subjects of Candidate's dissertations are not very different from the subjects of Doctor's dissertations (which is not surprising): in Candidate's dissertations, too, the focus is on problem solving and the history of mathematics education, on the development of creativity and the continuity of education, on teaching specific mathematical subjects, and the preparation of future teachers. The principal difference is in the scope and size of the studies: the aims of Doctor's dissertations are far more ambitious, the theoretical investigation must be far more profound and multifaceted (ideally, a Doctor must create a new theory), the experiment is larger and longer, the expected practical applications must be far more significant, and so on. At the same time, we should note that the chapters of Candidate's dissertations are quite expansive and contain discussions on theoretical concepts and categories and the existing literature, and also (usually) methodological and experimental sections. To repeat, Candidate's dissertations also require that the main results be published *prior to* the defense.

16 Conclusion

The long series of studies presented above, albeit very briefly, may be analyzed in a variety of ways. The first and perhaps most natural approach would be to think about the various concrete ideas that they contain, and about the ways in which these ideas might be applied and developed under different conditions (or, conversely, about the ways in which certain approaches popular in the West have become transformed in Russia — the whole panoply of ideas pertaining to "humanitarization" and "humanization," to give one example). This, of course, cannot be done within the scope of a single chapter.

Conversely, one may give further thought not to individual studies, but to the organization of scientific work as a whole, explicitly or implicitly comparing it with how matters stand in other countries. The observation that immediately comes to mind is that increasing centralization, control, the role of so-called accreditation, and so on, which many see as the best means of improving quality, in reality cannot guarantee high quality. Among the studies examined above, along with interesting and substantive works, we encountered a considerable number of patently weak works, which nonetheless have successfully passed through the multistage system of centralized control and evaluation.

Very weak (just like very strong) dissertations can probably be found without difficulty in any country. But both the strong and the weak aspects of dissertations to some extent exhibit distinctive features in different countries. The arguments demonstrating the effectiveness of some chosen approach, which were used in a number of the works discussed above, will not always appear convincing, for example to a Western reader; the reader might judge the system of argumentation to be biased and not objective. Reliance on theory, which we would describe as a positive feature in general, can also be excessive — the lists of thinkers who have influenced an author in the writing of various methodological recommendations often include major philosophers or mathematicians whose work is rather distant from the subject examined in these recommendations. The tendency to generalize and to engage in general, abstract reasoning, while commendable in principle, is often excessive; at times, the writing style begs to be parodied. Instead of saying that there are, for example, three quantitative attributes, some authors would invent something grandiose and convoluted, such as the element of a ternary relation defined over the set of real numbers.

Usually, a very large amount of work goes into both Russian Doctor's dissertations and Candidate's dissertations. In response to the question that students love to ask about how many pages they have to write, one may say that the number of pages that has to be written to obtain a Doctor's (and even a Candidate's) degree is indeed great. This refers not only to the dissertation itself but also to published papers which are required, and to methodological and educational materials that implement the author's approach. Here we come to what we would consider the main merit of Russian scientific works: they are works that, at least in terms of their subject matter, aim to improve teaching in schools.

Kilpatrick (2010), talking about a past period, has noted: "Much Soviet research in mathematics education took place in schools rather than laboratories, and it dealt with concepts from the school curriculum rather than artificial constructs — features that were especially attractive to U.S. researchers" (p. 361). While the situation in the West has somewhat changed in subsequent years, it would not be an exaggeration to say that in the West work that is specifically aimed at improving how children are taught — work on textbooks and programs — is largely (even if with important exceptions) a commercial, rather than a scientific, concern. As for scientific work, it is often focused on far narrower problems, which in our view pertain, strictly speaking, to psychology, not to mathematics education. The loss here is twofold: commercial textbooks lack deep methodological ideas (or sometimes *any* methodological ideas), while scientific works lack a sense of reality and practical applicability.

It is by no means necessary to agree with all of the methodological approaches developed by the authors of the works cited above. What is important, however, is that these authors feel the need for a connection — even if sometimes a purely nominal one — between their research and real schools and real colleges (however obscured this connection might be by complicated terminology). The sharp rise in the number of doctoral dissertations defended in recent decades was probably not accompanied by an improvement in their quality: the inflation in the significance of academic degrees that can be seen everywhere in the world is fully in force in Russia. Yet the orientation toward schools, and toward the teaching of mathematics in schools, endures. This fact sustains our interest in the development of Russian research on mathematics education.

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